

Introduction to *Knowledge, Number and Reality. Encounters with the Work of Keith Hossack* (London: Bloomsbury Academic 2022)

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Keith Hossack is a realist and a rationalist. His views are bold, controversial, unorthodox, sometimes outrageous, but always forcefully argued. He has published two monographs: *The Metaphysics of Knowledge* (2007) and *Knowledge and the Philosophy of Number: What Numbers Are and How They Are Known* (2020), together with a number of influential papers in philosophical journals. (See Chapter 1 for a full bibliography.) In this volume, we collect together new articles by his students and colleagues, many of whom have become his friends, engaging with all aspects of his work: metaphysics, epistemology, the philosophy of mind, logic and the philosophy of mathematics. Because Hossack's interests are so broad and systematic, this volume will be of interest not only to those who know his work, but to anyone engaging with the central questions in analytic philosophy.

This book grew out of the conference 'Reviving Rationalism. A Celebration of the Work of Keith Hossack', held in his honour on 26 November 2016 at Birkbeck College, University of London. It was organised by two of the editors and Simon Hewitt, one of the contributors to this volume. As Hossack was reducing his teaching duties at Birkbeck College, where he is a Reader in Philosophy, we thought it was a good time to celebrate his contribution to philosophy, as a thinker and teacher – not to mention, as *our* teacher. Thanks to a Mind Association Conference Grant and to Hallvard Lillehammer, who secured a grant from Birkbeck College, we held the conference. This book is a record of recent engagements with Hossack's work which have grown out of it.

We would like to thank all the contributors who have made this volume possible, and everyone whom we approached but who was prevented from contributing due to other commitments, our time frame or the theme of the volume. Apologies to all of Hossack's friends and colleagues we did not ask. The project proved very popular, and we are aware that many more could have contributed - at 14 contributions we needed to draw a line. Thank you also to the referees for Bloomsbury Press for their enthusiasm for our project and to Colleen Coalter, Becky Holland and their team for their support during the process of publication.¹

We divide up Hossack's oeuvre, somewhat artificially given the systematicity of this thought, into three themes: Realism, Knowledge, and Rationalism. Our introduction will follow this outline. Hossack takes much of his inspiration and starting point from Russell's writings, one of his philosophical heroes. In relation to Realism, we will pay particular attention to how Hossack's views interact with Russell's. Then we'll move on to explore the interaction between Hossack's work and the contributions to this volume.

The notion of realism central to Hossack's work is the doctrine that there are universals which exist independently of the particulars that instantiate them. Hossack adopts Russell's account of universals (Russell 1912: 145ff), according to which a universal is an aspect of resemblance (Hossack 2007: 34ff). Such an account of what a universal is may be uncontroversial. Philosophical controversy arises over the further question whether there are any universals and if so, what kind of things they are.

Hossack and Russell observe that some things really resemble each other: it is not just that they *appear* to someone to resemble each other, or that they are *perceived* as

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similar. Socrates and Plato resemble each other in being mortal. Their mortality is not a question of what someone may consider them to be. They really are mortal, no matter whether someone conceives them as such. If things resemble each other objectively, then, Russell and Hossack conclude, there must be universals. (See Hossack 2007, ch. 2.1, Hossack 2020, ch.1.) If two things really resemble each other in a certain respect, then they share the universal that is the respect or the way in which they resemble. Socrates and Plato both share the universal *mortality*, or they both instantiate it. Some universals are not instantiated by particulars but by other universals. Red and green resemble each other in being colours, so the universals *redness* and *greenness* instantiate the universal *colour*. The referents of the concepts 'square' and 'round' resemble each other in being shapes, so the universals *squareness* and *roundness* instantiate the universal *shape*.

The existence of universals, in turn, explains the nature of resemblance. Things resemble each other because they literally have something in common: the universal they both share. Maybe everything resembles anything in some aspect or other, or can be conceived to do so, but some things also fail to resemble each other in some respects. Wherever there are things that are not similar, there are aspects in which they do not resemble.² This, too, is explained by universals. If two things do not share a universal, they do not resemble each other in this respect. Although Socrates and Thrasymachus resemble each other in being mortal, they do not resemble each other in being wise. Socrates is wise. Thrasymachus is not wise. This is a matter of fact. But Thrasymachus resembles Meletus in

² One might be tempted to say 'whenever there are things that are different, there is an aspect in which they do not resemble'. But this requires a commitment to the claim that if *a* and *b* are different, then there is some property that they do not share. In conversation, Hossack has expressed reservations about this half of Leibniz' Law and pointed out that the desired effect may be had by the weaker claim relying only on dissimilarity.

this latter respect. Hossack draws a further conclusion: that there must be a universal *negation* (Hossack 2007: 62ff) that accounts for the failure of Thrasymachus and Meletus to instantiate the universal *wisdom*.

Negative facts, facts involving universals that are not instantiated by particulars or other universals, play a central role in Hossack's metaphysics, not least because the more precise account of the distinction between universals and particulars offered by Hossack in the development of his metaphysics is founded on the distinction between negative and positive facts, and his acceptance of the existence of both kinds. The acceptance of negative facts is repugnant to many metaphysicians. Russell famously observed that 'there is implanted in the human breast an almost unquenchable desire to find some way of avoiding the admission that negative facts are as ultimate as those that are positive.' (Russell 1919b, 4). He reports that his view that there are negative facts nearly provoked his students at Harvard to riot: 'The class would not hear of there being negative facts at all.' (Russell, 1919a, 42). Most metaphysicians believe that everything that exists is essentially positive. Russell, however, at least during the period of his thinking that led to *The Philosophy of Logical Atomism*, argued that negative facts cannot be avoided, and that any attempt to do so only brings them back in another guise. (See (Russell 1919a, 42ff) and (Russell 1919b, 4ff).)

Hossack, and Russell circa 1918, therefore share the unorthodox view that there are negative facts, but they differ strongly over their characterisation. According to Russell, the positive fact that *a* stands in relation *R* to *b* contains the relational universal *R* and the particulars *a* and *b*, while the negative fact that *a* does not stand in relation *S* to *b* also contains no more than the relational universal *S* and the particulars *a* and *b*: 'It must not be

supposed that the negative fact contains a constituent corresponding to the word “not.” It contains no more constituents than a positive fact of the correlative positive form. The difference between the two forms is ultimate and irreducible.’ (Russell 1919b, 4) While for Russell, positive and negative facts are not distinguished by their constituents, but only by their form, or the way the constituents are assembled to form the fact, Hossack, by contrast, accepts precisely what Russell denies, and explains negative facts in terms of the universal *negation*: it is the referent of the symbol for negation in logic. According to Hossack, the negative fact that *a* does not stand in relation *S* to *b* contains the universal *negation*, the relational universal *S* and the particulars *a* and *b*.

Negative facts are also important in other areas of Hossack’s metaphysics. He likes to point out that omissions have causal powers: you forgot to water the plants and they died. The omission to water the plants is a negative fact: the plants were not watered. This negative fact causes the death of the plants. Likewise, a bridge may collapse because of the negative fact that it did not have enough rivets.

Once an ontology of universals is accepted, other considerations than aspects of resemblance may come into play in tackling the question which universals there are. It may be that not every resemblance requires its own universal, but it suffices if every aspect of resemblance is explained in terms of some universals. There need not be a universal *non-wisdom*, but it suffices that there are the universals *negation* and *wisdom* to explain the resemblance between Thrasymachus and Meletus in this respect. It is worth drawing a distinction between properties and universals. Not every property need be a universal: although there is the property *non-wisdom*, there may not be such a universal. Some properties are universals, others are more accurately described as compounds of

universals, such as the property *non-wisdom*, which is a compound of the universal *negation* and (the universal or property) *wisdom*.

Some properties are not shared by some objects: one object has a property that the other doesn't. Some properties are even had by nothing at all. Nothing has the property *unicorn*. Worse still, some properties could not be had by anything. Nothing could have the property *non-self-identical*. But they are nonetheless properties, according to Hossack (2007: 58). Hence, according to him, there are actually and even necessarily uninstantiated properties.

Hossack also accepts that universals may exist even if they are uninstantiated. For instance, according to Hossack there are logical and mathematical universals. But nothing in the world, as it is, instantiates the geometrical universal *roundness*. There are things that more or less approach being round in the perfect, geometrical sense, but there is nothing such that every point of its circumference is equidistant to its centre. Whether there may also be necessarily uninstantiated universals would appear to be a further question.

This goes against a popular view going back to Aristotle that only those properties and universals exist which are instantiated. Another popular view is that properties of spatial objects are located where their instances are. This view, too, is not one Hossack agrees with: if uninstantiated properties exist, they cannot be located where their instances are, as there are none.

Many philosophers agree that there are properties or universals, but not many will follow Hossack to the extreme realism he is prepared to accept, where even *negation* is a universal, and where properties and possibly even universals may be necessarily

uninstantiated. Hossack is not afraid of drawing controversial conclusions from apparently self-evident first principles.

As we will see, some passages of *The Metaphysics of Knowledge* may sound as if Hossack tends towards a view according to which universals are abundant: a universal is everything that is not a particular, and thus it would appear that any meaningful predicate corresponds to a universal. But a clear counterexample is Hossack's rejection of the view that the predicate 'true', while meaningful, stands for a universal (Hossack 2007: 88). Truth is defined as correspondence to the facts, but that my belief corresponds to the facts is not a property it has, and there is no universal *truth*. Hossack draws the consequence that there is no such link between true *beliefs* and the facts to which they correspond as there is between mind and world when a fact is *known*.

Elements of a sparse view of universals are more prominent in *The Metaphysics of Knowledge*, and this is how Hossack intends to be understood. According to the sparse view, only some meaningful predicates correspond to universals, namely those that latch on to fundamental features of reality. Hossack accepts Socrates' account of a good definition given by Plato in the *Phaedrus* (265e-266a): to give a good definition is to 'cut up each kind according to its natural joints'. 'Plato's claim is that scientific progress requires the discovery of the principles of classification that divide things according to their real resemblances and real differences. The laws of a science connect the properties that are of classificatory importance; thus the very idea of a law of nature is inseparably connected with real or objective resemblance.' (Hossack 2007: 35f) Which universals there are is a question of what the world is ultimately like and what laws govern it. They are discovered by us in our attempts to produce optimal descriptions of the world.

In Hossack's view (2007, ch.2.1, 2020, ch.1), realism about universals, contrary to its nominalist rivals, can explain the difference between a natural class and a merely miscellaneous collection. Hossack argues for a realist position about mathematical entities too. In his view, numbers are universals. And precisely because of this, our knowledge of them is no more mysterious than our knowledge of any other universals. In fact, the source of epistemological puzzlements about the existence of numbers lies in the thesis that numbers are abstract *objects*. Hossack argues that this thesis traps us in the skeptical conclusion that we cannot have knowledge of numbers and number-theoretic propositions. The right conclusion to be drawn from this sort of skepticism, Hossack claims, is that numbers are not particulars. They are universals.

Abundant or sparse, Hossack's account of universals locates them in the vicinity of Plato's Realm of the Forms. His realism can be described as a species of Platonism, and Platonist themes abound in his work. Russell himself describes his theory of universals as 'largely Plato's, with merely such modifications as time has shown to be necessary' (Russell 1912: 142f). In conversation, Hossack likes to emphasise that Plato and Russell wrestled with the same fundamental problems, as does he, following in their footsteps.

Though fundamental to Hossack's metaphysics, *universal* and *particular* are not its most fundamental theoretical notions. These are the notions of *knowledge* and of *fact*. According to Hossack, knowledge is a metaphysically and conceptually primitive notion. *The Metaphysics of Knowledge* defends this view in great detail and applies it to a variety of philosophical issues in metaphysics, epistemology and the philosophy of mind in a demonstration of its strength and versatility. The notion of fact is required as that which is

known, as one of the relata of the primitive notion of knowledge, and as that in which universals and particulars are combined.

A primitive notion is one that cannot be defined or analysed any further. This does not mean that we cannot say anything informative or illuminating about it. For instance, being a primitive notion, knowledge is not a species of belief. In particular, if knowledge is primitive, then it is not justified true belief, as in the definition of knowledge first propounded in Plato's *Meno* (98a), though already shown to be problematic in the *Theaetetus* (201cff). Hossack subscribes to a causal thesis about the relation between knowledge and belief and, because knowledge is primitive, he rejects the commonly held 'constitutive theses'. According to Hossack, knowledge is caused by beliefs and the exercise of mental faculties, but it is not constituted by them. In particular, knowledge is not constituted by justified true belief plus something or other. The primitive notion of knowledge Hossack has in mind is a direct relation of awareness between a mind and that which is known. This is a relation between mind and world that is not mediated by ideas or thoughts or mental representations. The mind that knows stands in direct contact with that which is known.

Consequently, the primitive notion of knowledge is not a propositional attitude either, and the notion of knowledge on which Hossack's view is founded must be distinguished from the complex relation expressed by the sentences that are used in knowledge attributions. According to Hossack, in the attribution 'X knows that *s*', the sentence *s* serves to report not only the fact of which X is aware, but also the content of the mental event in virtue of which X apprehends the fact in question. Hossack therefore rejects the theory that the attribution 'X knows that *s*' expresses a propositional attitude, for an

attitude is a relation between a mind and a mode of presentation, or in alternative terminology, a content, a Fregean Thought, or a proposition. Therefore if the attribution 'X knows that *s*' expressed only a propositional attitude, X would stand only in a mediated relation to the fact that *s*: the relation between the mind and the fact would be mediated by the mode of presentation and it would only be to the latter that the mind would stand in a direct relation.³

Here another Hossackian theme emerges: the refutation of scepticism, also a concern he shares with Russell. If knowledge were merely a species of belief, I could be in the same state of mind as someone deceived by an evil demon. If beliefs or ideas or propositions or other modes of presentation mediated between the mind and the world, and knowledge arose when these mediators in turn relate correctly to the world, i.e. are true and arrived at in suitable fashion, then the possibility contemplated by scepticism may actually be the case. The sceptic posits that I could be in the same epistemic state as someone who is deceived by an evil demon. Hossack responds that I could not be in such a state, as knowledge is not a species of belief, and if I know, then I stand in the knowledge relation to facts in the world, and hence I am not deceived by an evil demon. (Hossack 2007: 16) Believing and knowing may be phenomenologically indistinguishable to the mind that believes and knows, but they are distinct nonetheless: one brings the mind into direct contact with the facts, the other does not. Thus this kind of scepticism is built on a false assumption.

That which is known is a fact. Hossack's work relies on a substantial theory of facts. 'Realism, the theory of universals, needs to be completed by the addition of a theory of

³ In this and the previous paragraph the authors have relied on Hossack's explanation of his views in private correspondence.

facts.' (Hossack 2007: 45) In explicating the notion of fact, Hossack also characterises the kind of relation that knowledge is. A fact is typically a combination of universals and particulars, such as that Socrates is wise: the fact combines the universal *wisdom* and the particular Socrates. Furthermore, facts combine universals and particulars in orderly fashion. There is a difference between the fact that Romeo loves Juliet and the fact that Juliet loves Romeo: one could obtain without the other.

The notion of fact is a fundamental metaphysical primitive of Hossack's philosophy, and with the facts comes the combination of universals and particulars into facts. Hossack argues that realism and the theory of facts go hand in hand: a satisfactory theory of universals requires the theory of facts, and a satisfactory theory of facts requires realism. (Hossack 2007: 34) The notions of universal and particular are explicated beyond their initial characterisation, as aspects of resemblance, in terms of how they combine to form facts.

According to Hossack, 'realism takes the highest categories to be particular and universal: it claims that without exception, everything that exists falls into one category or the other' (Hossack 2007: 34). It might be worth pointing out a consequence. Bradley's Regress is the problem that, if instantiation were a universal, then for *a* to instantiate *B*, there should be a further kind of instantiation which is instantiated by *a*, *B*, and the universal *instantiation*. Clearly, the same considerations apply to this further kind of instantiation, and so on *ad infinitum*. And clearly, they also apply to the combination relation. To avoid Bradley's Regress, the combination relation must be something other than a universal. Combination certainly is not a particular. Consequently, the combination relation falls outside the categories of universals and particulars. It is syncategorematic, as

the scholastics might say, and cannot properly be said to exist. Facts are the rock bottom of Hossack's metaphysics: in those that exist, universals and particulars are combined. Instantiation of a universal by a particular is explained in terms of facts, not conversely.

Russell concluded from his explanation of the nature of universals and the objective resemblance of things in certain respects that there must be universals. It is a further question whether there are also particulars. Russell admitted the possibility that there are no particulars. (Russell 1911: 10ff) To tackle this issue, we need a definition of 'particular'. So far, we had a heuristic account of the nature of universals, but a sharper definition of this notion is also desirable and available on the basis of the definition of 'particular'. Hossack's definition of 'particular' and 'universal' relies on the existence of negative facts, facts that contain the universal *negation* in prominent position. The latter is the referent of the symbol for 'not' in formal logic. It works like any other universal, in that the combination relation forms facts from it together with other things. A Russellian proposition is defined as a number of ordered universals and possibly particulars, so that either they combine into a fact, or the universal *negation* and they combine into a fact. (Hossack 2007: 62ff) Particulars and universals can then be defined in a strikingly Aristotelian fashion. Aristotle writes in the *Categories* that 'a substance is that which is neither said of a substance nor in a substance' (2a13). According to Hossack, 'a particular is anything that occupies "predicate position" in no Russellian proposition; a universal is anything that is not a particular' (Hossack 2007: 66). It follows that facts are particulars.

It is from here that the appearance of abundance in Hossack's account of universals stems. If a universal is anything that is not a particular, and a particular is anything that cannot occupy predicate position in a proposition, then anything that can occupy predicate

position in a proposition is a universal, and so it may appear as if any predicate that occurs in a sentence that expresses a proposition refers to a universal. Which propositions there are, however, and hence which sentences express propositions, depends on which facts there are, and which facts there are is determined by nature. If nature is sparse, then so are the universals, and Hossack, as we saw, follows Plato in this respect.

According to Russell, all knowledge must involve universals. It is a further question whether knowledge may also be had of particulars. On Hossack's account, this question is settled decisively. The fundamental epistemological relation is the relation of knowledge between a mind and a fact. Facts belong in the category of particulars, and so do minds. Hence knowledge is a relation between two particulars, one of which is a mind, the other a fact, in that order. Thus, some particulars are known, namely the facts. In *The Philosophy of Logical Atomism*, Russell famously argued for the view that while facts could be asserted to obtain, they could not be named (Russell (1918): 507f): facts are *sui generis* and unlike universals or particulars. Hossack disagrees: as facts are particulars, nothing prevents them from being named.

Knowledge defines the nature of mind. A mind is something that stands in the knowledge relation to some fact at some time. It is of the essence of a mind that it knows something. (Hossack 2007: 169) One should maybe add, to allow Socrates or a sceptic to have a mind, that a mind need not know that it knows something. Whether the mental constitutes a special substance, as Descartes held, or is identical to something physical, such as a person's brain, is left open by this account of the mark of the mental. Cartesian souls and brains are equally candidates for the relata of the primitive knowledge relation.

However, insofar as knowledge is not a relation known to physics, the physicalist option may only be open to non-reductive physicalists or property dualists.

Central to the philosophy of mind are the notions of consciousness and conscious perception. Hossack explains both in terms of the primitive notion of knowledge. 'Perception is the faculty that gives us knowledge of facts about our environment' (Hossack 2007: 14). Hossack's theory of consciousness builds on an identity thesis which he derives from the Scottish philosopher Thomas Reid: 'A mental act is conscious if and only if it is identical with knowledge of the quale it itself has' (Hossack 2007: 169). Qualia are often explained as the 'subjective character' of some mental states, in particular of perceptions. Hossack explains qualia also by an identity thesis and a further appeal to the primitive notion of knowledge. 'One's pain is identical with one's awareness of the qualitative character of the pain' (Hossack 2007: 185). A conscious perception is identical with knowledge, or awareness, of the quale of the perception. A quale is a property of an experience. According to Hossack, a quale is a universal Q such that, for any x , if x instantiates Q , then x is identical to knowledge of x . (Hossack 2007: 186) Thus conscious perception is a form of knowledge, but the crux of the matter lies in that which is known. Conscious perceptions instantiate qualia, and an instantiation of a quale is identical with knowledge of the instantiation. Suppose Frege smells the scent of violets. This event is the same as Frege's knowing that he smells the scent of violets. His experience has a certain quale, that of smelling of violets, and Frege knows that it does:

e = Frege smells the scent of violets = Frege knows that e has the quale of smelling of violets
= Frege knows that Frege's smelling the scent of violets has the quale of smelling of violets.

In general:

X perceives $p = X$ knows that X 's perceiving that p has quale Q .

The perception has the fact that is the perceiving of the percept by the perceiver as a constituent. Thus the fact that X perceives p occurs as a constituent of a constituent of itself, as it instantiates a quale and it is known by X that it does so.

Hossack's theory of consciousness appeals to facts that contain themselves as constituents of some of their constituents. Hossack draws a parallel between this phenomenon with one found in non-standard set theories. In the terminology of set theory, the facts that constitute consciousness are not well-founded. But as non-wellfounded set theory is a respectable mathematical theory, and consistent if standard set theory is, Hossack argues that non-wellfoundedness is not objectionable in the theory of facts that explain consciousness either.

As this aspect of Hossack's philosophy of mind is not as well known as his metaphysics, it may not go amiss to spell out some of the basics of non-wellfounded set theory on which his account of consciousness draws. There is a certain discrepancy in the names given to the axioms of Zermelo-Fraenkel set theory, but with adjustments of terminology, what is to follow may be found in any standard text book on set theory, such as those of Fraenkel, Bar-Hillel and Levy (1973), Enderton (1977) or Mendelson (2015).⁴

⁴ Nils Kürbis has profited from discussions with Neil Barton in getting to grips with non-wellfounded set theory and would like to thank him here.

What is commonly referred to as the Axiom of Regularity states that every non-empty set has a member that is disjoint from it: $\forall x(x \neq \emptyset \rightarrow \exists y (y \in x \wedge y \cap x = \emptyset))$. A consequence of the axiom is that no set contains itself. For suppose $A \in A$, and consider its singleton set $\{A\}$, which exists by the Axiom of Pairing. It is non-empty, so by Regularity it has an element that is disjoint from it; i.e. for some $y, y \in \{A\} \wedge y \cap \{A\} = \emptyset$. But the only element of $\{A\}$ is A , hence $y = A$. But $A \cap \{A\}$ contains A , hence is not \emptyset . Contradiction. More generally, suppose there is a finite sequence of sets $A_1 \in A_n \in A_{n-1} \dots \in A_2 \in A_1$. Then by the Axiom of Separation, there is a set $A = \{A_n, A_{n-1} \dots A_2, A_1\}$ containing all and only these sets. A is non-empty, and hence by Regularity it contains an element disjoint from it. But $A_n \cap \{A\}$ contains A_1 , and for any $i < n$, A_i contains A_{i+1} , hence the intersection of A with any of its elements is non-empty, contradicting Regularity. Hence, if facts were like the ordered n -tuples of set theory, Zermelo-Fraenkel set theory would exclude the existence of facts of the kind on which Hossack's theory of consciousness relies. Hossack's facts, therefore, are not such ordered n -tuples.

What is commonly referred to as the Axiom of Foundation states that there is no function that has as its domain the set ω of the natural numbers and as its range a sequence of sets $\sigma_0, \sigma_1 \dots$ such that every $i, \sigma_{i+1} \in \sigma_i$. The Axiom of Foundation implies the Axiom of Regularity, and in the presence of the Axiom of Dependent Choice, the Axiom of Regularity implies the Axiom of Foundation. A consequence of the Axiom of Foundation is that there are no infinite descending sequences of sets standing in the membership relation. Following down the elements of the elements of the elements ... of a set A , we reach an end after finitely many steps. Any sequence of members of any set A such that $\dots \in x_3 \in x_2 \in x_1 \in A$ ends after a finite number of steps with the empty set. Thus, in Zermelo-Fraenkel set

theory, the membership relation is *well-founded*. In non-wellfounded set theory, which rejects the Axioms of Foundation and Regularity, it is not. Infinitely descending and circular sequences of sets standing in the membership relation are permitted. But notice that, just as already finite circles of membership are excluded in Zermelo-Fraenkel set theory, non-wellfoundedness does not always induce infinite chains or infinite circles of membership: if $A \in A$ and nothing else is, the chain is only finite. Some non-wellfounded sets have only finitely many elements and are perfectly finite objects. This important detail forestalls the objection that Hossack's theory of consciousness should be rejected because it induces some kind of 'infinite regress' in the non-wellfounded facts to which it appeals. This is not the case. The facts on which Hossack's theory builds may be as finite as any well-founded facts.

Rationalism is the doctrine that there are truths about the world that can be discovered by pure reason alone, that there is *a priori* knowledge about mind-independent reality. An unwavering belief in the power of pure reason is characteristic of Hossack's approach to philosophy. If pure reason leads us to conclusions that are unorthodox, counterintuitive or uncommon, then so much the worse for orthodoxy, intuition or common opinion. It is also typical of Hossack that he is capable of approaching a discussion afresh, throwing new light on it by looking at it in rationalist fashion from first principles and following a line of thought where it leads.

One example concerns Hossack's views on the relation between necessity and the *a priori*. According to Hossack, the truths that are knowable *a priori* are exactly those that are necessary. This goes contrary to the received view, propounded in Kripke's *Naming and Necessity*, that there are necessary truths which can only be known *a posteriori*, such as that

the Morning Star is identical to the Evening Star or that water is H₂O. Hossack's line of argument to establish that the necessary and the *a priori* co-incide is elaborate and intricate, and we refer the reader to Hossack's writing on the matter (Hossack 2007, ch. 4), which provide a refreshing challenge to received opinion.

In the philosophy of mathematics, Hossack (2020) shows how one can have an *a priori* science of mathematics. His philosophy of mathematics revives the Aristotelian thesis that numbers are magnitudes, and thus a kind of property. More precisely, a magnitude is a property that is shared by equivalent quantities. For example, the natural number 2 is a property that is shared by all pairs. This simplified version of the Magnitude Thesis plays a substantial role in Hossack's project.

Hossack starts from the Aristotelian thesis that the subjects of a judgment are divided into the categories of *individual* and *quantity*. Examples of quantity considered by Hossack are pluralities (some books), continua (such as a stretch of space and time), or series (such as Plato and Socrates in that order). According to Hossack's metaphysics of mathematics, the natural numbers are properties of pluralities, the positive real numbers are properties of continua, and the ordinal numbers are properties of series. These different kinds of numbers reflect different categories of quantity. Hossack's project thus runs in an anti-Fregean direction. According to Frege and his neo-Fregean followers, numbers are particular objects, which are the referents of semantically singular terms.

Hossack pursues the Aristotelian line of thought that a quantity is what is divisible into 'two or more constituent parts' (Aristotle, *Metaphysics* 1020a7). Thus, quantities have a mereological, or part-whole, structure. Hossack shows that the axioms for the natural and

real numbers, and also for the ordinal numbers, can all be deduced from mereology plus the *a priori* axioms that Euclid calls the Common Notions:

1. Quantities which are equal to the same quantity are also equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Quantities which 'coincide' with one another are equal to one another.
5. The whole is greater than the part.⁵

Thus, quantities satisfy a notion of equality. For example, two pluralities are equal just in case they can be put in one-one correspondence. In Chapter 4 of his (2020), Hossack discusses Leśniewski's mereology and its axiomatization by Tarski. He argues that W. V. Quine, Peter Simons, and David Lewis have interpreted mereology as a theory about 'individuals', and thus the axioms of mereology, as interpreted by them, cannot be known *a priori*. According to Hossack's alternative interpretation, mereology is a theory about 'items in the category of quantity' (Hossack, 2020, p. 51).

Hossack puts forward nine axioms of mereology each of which is known *a priori* when interpreted as a law of the logic of pluralities. Likewise, he puts forward nine axioms of mereology each of which is known *a priori* when interpreted as a law of the logic of continua. He shows that eight of the nine axioms are common to pluralities and continua. Hossack proves that these Common Axioms are deductively equivalent to Tarski's and

⁵ Quoted from Hossack (2020, p.4)

Simons' axiom systems, and concludes that this technical result mandates an interpretation of mereology as the 'pure *a priori* logic of quantity' (2020, p. 7).

The upshot of Hossack's mathematical project is that our epistemology of number neither relies on empirical evidence nor on set-theoretical authority. It does not appeal to Fregean abstractionist resources, either. Hossack argues that Euclid's Common Notions and the fundamental laws of mereology are evidently *a priori*, and so, "by reason alone", we can arrive at our knowledge of number and of number-theoretic truths.

Contributions

The first Chapter is a short statement of **Keith Hossack's** philosophical views which he wrote specifically for this volume, reflecting the main theses and developments of his thought. We will let it speak for itself.

Mark Sainsbury's contribution, 'Confronting Facts: on Hossack's *The Metaphysics of Knowledge*' (Chapter 2), concerns a central theme of Hossack's statement: *relation dualism*. This is the doctrine that there is an irreducible, primitive term in the theory of mind, which refers to something that is both mental and a relation. Hossack holds this doctrine to be true and calls this relation 'awareness' or 'knowledge of'. He takes the notion to be unanalysable and metaphysically fundamental.

Sainsbury does not take issue with the concept of knowledge being unanalysable, but finds it 'harder to say what metaphysical fundamentality amounts to' (p.39). One natural way of understanding it is in terms of the directness or simplicity of the way the

subject relates to the world. In a memorable example of Hossack's, this is the simple and direct way in which an amoeba is aware of the light.

According to Sainsbury, there is a tension in Hossack's views on knowledge. Hossack holds both (a) that knowledge is metaphysically fundamental, and (b) that knowledge is caused by belief under suitable conditions. That knowledge is caused by belief, where belief is a relation to a proposition, would make knowledge appear non-fundamental. For, firstly, any knowledge-of – the direct, fundamental sort of knowledge – would come causally downstream of knowledge-that – the indirect, non-fundamental sort. And so knowledge-of would be dependent on knowledge-that. Whereas, Sainsbury takes it, that which is metaphysically fundamental is that which is non-dependent. Secondly, knowledge-that is mediated by a proposition – but how can knowledge-of, which is simple and direct, be caused by knowledge-that, which is a mediated relation to reality?

Sainsbury offers Hossack a friendly amendment: a way of giving knowledge-of a 'special metaphysical and explanatory role', a role he takes Hossack, in any case, to be committed to. Sainsbury does so by working out the view that perceptual experience can be a 'distinctive source of knowledge-of which is not caused by knowledge-that' (p.40). According to his theory, (i) perceiving is gaining knowledge-of tropes without any specific knowledge-that, and (ii) non-inferential cognitive processes can select 'one of possibly many ways of conceptualizing that input to deliver knowledge-that' (p.43). This is, roughly, how knowledge-of can yield knowledge-that, in the case of perception.

Sainsbury takes this to explain how knowledge-of – something which is direct, and so non-propositional and obtained non-inferentially – can be a source of knowledge-that, which is propositional, and so can play a role in inference. He takes an important merit of

his theory to be its ability to make sense of our experience of perception – as distinct from testimony – which we express largely through metaphors: open, vivid, rich, coercive, immediate, present, simple and direct.

For Hossack, knowledge consists in a relation between a mind and a fact, and a fact always contains some universal. Hossack's realism about universals is a kind of Platonism. His epistemology resembles a traditional reading of Plato's, on which knowledge has as its subject matter the Forms. In her paper 'Who Knows' (Chapter 3), **MM McCabe** argues that there are significant differences between the approaches of Plato and Hossack, and contrasts Hossack's account with an epistemology developed from Plato's dialogues, in particular the *Republic*. In a nutshell, on Hossack's account, knowledge may be had of a very small number of facts and by a mind just happening to stand 'passively' in the right kind of relation to facts, whereas for Plato, knowledge is always of a large range of facts and can only be acquired actively.

On Hossack's view, it should be possible to know only very few facts. If I stand in the right (or wrong, depending on how you look at it) causal relations to the world, I may know that Jamina is a pygmy hippopotamus and not much else. To know this fact I also need to know a few others, such as in which zoo I saw her and that she is a fairly large, round animal with greyish, brownish skin mostly lying in her pond or eating large amounts of hay. But it certainly does not require much knowledge about pygmy hippos, not even that she is a member of a species that occurs in the wild only in West Africa. Furthermore, it is not much of my doing that I came to know that Jamina is a pygmy hippo: it sufficed that we both were in the right place at the right time and causally connected in the right way.

This account of knowledge contrasts sharply with Plato's, explains McCabe. According to Plato, knowledge is systematic, it is a virtue, it is the foundation of wisdom and it is good. Knowledge encompasses entire domains of what is known. Thus it can only be acquired in a laborious way through long years of training and study. McCabe's Platonist epistemology sees knowledge as akin to understanding, which brings with it many related items of knowledge at once. Understanding is the result of learning, and learning is a process in which an agent engages. Being a virtue, to know requires a certain kind of agent. Just as 'doing the right thing' does not make you virtuous, 'having the right state of mind' does not make you knowledgeable. Being the foundation of wisdom, knowledge is hard to acquire. The value of knowledge lies in the development of the knower. For Plato, to know requires conditions internal to the agent that Hossack's externalism neglects, argues McCabe. It requires an active process in which the knower engages, not the passive one of a mind standing in the right causal relations to facts. The path to knowledge is long, arduous and essential to what it is to know. It also requires an appreciation of one's own epistemic state: to know, in the words the god at Delphi, you need to know yourself.

Despite their differences, there is an illuminating agreement between Hossack and McCabe. Both agree that knowledge is not a species of belief. Coming to know is moving from *doxa* (belief) to *epistêmê* (knowledge) which are different powers or faculties, as McCabe puts it: they are two different states of mind.

Scott Sturgeon's paper 'Knowledge-first Epistemology and the Input Problem' (Chapter 4) takes a very different approach to McCabe and represents epistemology in the most abstract, theoretical way possible. Sturgeon describes what he

calls the 'Input Problem', from which he takes a number of epistemologies to suffer, including Hossack's knowledge-first epistemology.

Sturgeon outlines what he calls the 'full-dress theory of epistemic rationality' which is composed of four parts: a theory of epistemic targets (which says what is subject to epistemic evaluation), a theory of evidential information (which specifies which pieces of information count as evidence for a situated agent), a theory of evidential support (which explains how a situated agent comes to possess particular bodies of evidence), and a conversion theory (which says how agents should configure their epistemic targets in light of their overall situation).

Sturgeon maps out the logical space open to theories of evidential information and theories of evidential support. In both cases, the theory can be characterised in terms of whether it accepts or rejects each of three components; and in both cases, only six of the eight resulting options are viable. For theories of evidential information, these components correspond to answers 'Yes' or 'No' to the questions whether that from which evidence arises has propositional content, whether it supervenes on the mental, and whether evidence is veridical. For theories of evidential support, the questions are whether an agent's evidential support may be had *a priori*, whether there can be more or less of it, and whether it is absolute or relative to an agent.

Under certain plausible assumptions about the relation between knowledge, evidence and credence, Sturgeon argues that the credence one should afford to anything one knows must be certainty. But this, he points out, is absurd: an agent may know without being certain, and indeed, knowledge may even preclude certainty, due to the fallibility of agents or sources of knowledge. That is to say, we may know things without being sure.

Indeed, in many cases we know things and *ought not* to be sure of them, because we know them on the basis of sources we know to be suspect.

This is what Sturgeon calls 'the Input Problem'. It poses a theoretical difficulty for any epistemology which endorses the idea that evidence comes in bits of information, i.e. what Sturgeon calls 'alethic chunks' offered by sources of information. And for Sturgeon, this includes Hossack's knowledge-first epistemology. He goes on to investigate plausible responses to the Input Problem for a knowledge-first epistemology. This will mean rejecting one of the assumptions that generates the trouble: (a) that knowledge yields pieces of evidence, or (b) that pieces of evidence are always maximally supported by total evidence, or (c) that credence for a claim should always match that claim's evidential support on the total evidence.

In his paper 'Perceiving X = Consciousness of Perceiving X. Hossack and Brentano on the Identity Thesis' (Chapter 5), **Mark Textor** discusses Hossack's theory of consciousness with attention to a thesis of Brentano's and Hossack's (2006) reading thereof. As Textor touches upon aspects of Hossack's philosophy not covered by other papers in this volume, we'll devote a little more space to its discussion.

Hossack, Reid and Brentano agree on several points: that in conscious perception, knowledge of the perception and the perception necessarily coincide; that this must be recognised and explained by any satisfactory theory of consciousness; and that any theory that allows for them to come apart must be rejected. If I know that I am in pain or am aware of being in pain, I'm in pain. It is the converse that is striking: if I'm in pain, I have immediate and direct knowledge of the pain. The thesis that consciousness of perception and perception are identical immediately accounts for this remarkable feature. There are

not two events that always occur at the same time. There is only one. Conversely, explains Textor, Brentano observes that if the consciousness of a perception and the perception were distinct, the immediate and infallible knowledge we have of our own perceptions would no longer be explicable and, in fact, might fail: if there were two distinct events, each may occur without the other. Thus, Brentano begins his investigation with the epistemology of conscious perception, and its metaphysics must follow the epistemology.

According to Brentano's identity thesis, the mark of conscious perception is that the perception of an object is identical to the perception of the perception of the object. Conscious perception is perception of perception, but perception of perception just is perception. Hossack rejects Brentano's version of the identity thesis, because on his reading of Brentano, he cannot have held that the perception and the perception of the perception are literally the same event. An event, according to Hossack, is a temporal fact. A fact consists in the instantiation of universals by particulars or other universals. An event is, therefore, an instantiation of universals by particulars or universals at a time. Facts are identical if their constituents are identical. According to Hossack's reading of Brentano, a conscious mental event is identical to an event that contains itself as a constituent. But this is impossible, according to Hossack's criterion of identity for events. Consider:

X perceives p = X perceives X 's perception of p .

The event on the left-hand side of = contains the perceiver, the percept and perceiving: the perceiver and the percept (in that order) instantiate the universal *perceives* at the time of the perception. The event on the right of = contains the perceiver and the event on the left

of =. But the percept p is not identical to X 's perception of p . Hence the two events are not identical.

On Hossack's scheme, by contrast, an event that is a conscious mental act does not contain itself as a constituent directly, but only as a constituent of one of its constituents. According to Hossack, a conscious perception is an event e that is identical to X 's knowing that e instantiates a quale Q . Then e has as a constituent, not e itself, but the fact that e is Q . It is the latter that has e as a constituent, and although a certain circularity arises, this does not contradict Hossack's criterion of identity of facts (Hossack 2007: 185). The 'circularity' is the kind one encounters in non-wellfounded set theory and, as such, Hossack suggests, it is unproblematic.

To round off the discussion of non-wellfounded set theory, let us go through a detail of non-wellfounded set theory mentioned by Textor. Textor refers to the simplest non-wellfounded set as the one that is identical to its own singleton, i.e. the set A such that $A = \{A\}$. It follows from two basic axioms of set theory, the Axioms of Extensionality and Pairing, and first-order logic that A contains only itself. The Axiom of Extensionality states that if sets A and B contain the same elements, then they are identical. The Axiom of Pairing states that for any set A and B , there is a set containing just them, i.e. the pair set $\{A, B\}$. The case where A and B are identical guarantees the existence of the singleton set $\{A\}$ of any set A , i.e. the set containing A and nothing else:

$$(1) \forall x . x \in \{A\} \leftrightarrow x = A$$

We show that A is the sole element of itself if and only if A is identical to its own singleton.

For the left to right direction, suppose A is the sole element of itself:

$$(2) \forall x . x \in A \leftrightarrow x = A$$

So if $y \in A$, then $y = A$, hence y is an element of the singleton of A , i.e. $y \in \{A\}$. Conversely, if $y \in \{A\}$, then $y = A$, hence by (2) $y \in A$. Hence by the Axiom of Extensionality, $A = \{A\}$, and A is identical to its own singleton. Even more briefly, it follows from (1) and (2) by replacement of equivalents that $\forall x . x \in A \leftrightarrow x \in \{A\}$, and so by Extensionality $A = \{A\}$. For the right to left direction, suppose A is identical to its own singleton, i.e. $A = \{A\}$. Then by Leibniz' Law

$$(3) \forall x . x \in A \leftrightarrow x \in \{A\}.$$

(2) follows from (1) and (3) by replacement of equivalents, and A is the sole element of itself. Having appealed only to Extensionality and Pairing, $(\forall x . x \in A \leftrightarrow x = A) \leftrightarrow A = \{A\}$ holds in Zermelo-Fraenkel Set Theory, but both sides of the principal biconditional are false, as the Axiom of Foundation precludes the existence of a set that is its own sole member and that is identical to its own singleton.

Textor defends Brentano against Hossack's criticism, and argues that their views are closer than might appear. Textor recommends, however, not identifying mental acts, as understood by Brentano, with Hossack's ordered universals, particulars and times. It is worth noting that Hossack adopts his account of events because 'it fits conveniently with [his] metaphysics of facts' (Hossack 2007: 102): events are just another kind of fact.

Another popular account, namely Davidson's, according to which an event is a particular of a more mundane kind, may be better suited to Brentano's metaphysics of mental acts. Further considerations may convince Hossack to adopt it instead.

Textor's defence of Brentano appeals to his most famous thesis: intentionality is the mark of the mental. Mental acts are distinguished from everything else by being directed upon objects. Textor combines it with Hossack's work on plural reference to shed light on Brentano's account of consciousness. Plural reference occurs if an expression refers to more than one object, to a plurality considered as several, not as a unified collection. When I say 'The geese are noisy', I'm not saying that the pair set containing the two Egyptian geese outside my window are noisy. That is an abstract object and accordingly very quiet. It is the geese who are noisy, and 'they' refers to both of them together, as a plurality.

Textor explains that Brentano's mark of the mental should not be understood as if each mental act is directed upon only one object. Mental acts can be directed upon pluralities, just as words can refer to them. In fact, conscious mental acts are always directed upon more than one object, as they are always directed upon the acts themselves. A conscious mental act, such as smelling the scent of violets, is directed towards itself as well as towards some other object, in this case the violets. According to Textor's defence of Brentano, although in conscious perception, the perception and the perception of the perception are identical, there are two ways of conceptualising that one mental event, once as a perception and once as a perception of a perception. Thus, the two sentences:

- (1) Frege smells the scent of violets.
- (2) Frege is conscious of the smell of violets.

constitute two different descriptions of the same event. The plurality of the objects towards which the mental act is directed – the objects of the perception as well as the perception itself – underlies the different ways of describing it.

Hossack's theory of knowledge, as we have noted above, takes knowledge to be an unanalysable and metaphysically fundamental relation between minds and facts. And one of the many phenomena that can be understood in terms of knowledge is modality. **Bernhard Weiss's** paper 'Facts, Knowledge and Knowledge of Facts' (Chapter 6) puts pressure on each part of Hossack's theory. First on its ontology of facts; second, on the evidence for treating knowledge as a relation to a fact, and third, on the use of knowledge to explain modality.

First, ontology. What is known, in Hossack's theory, are facts. While facts are metaphysically basic, we can comprehend them through a theory of vectors and combination. Vectors are any entities taken in a particular order. Hossack defines a notion of sense for vectors: the entities of a vector make sense if and only if there is either a fact combining them or there is a negative fact combining them. So combination is a crucial relation in Hossack's understanding of facts. Weiss argues that this ontology of facts requires a *singular* combinatorial relation, whereas a closer look reveals a plurality of relations which relate entities in a vector such that they make sense. Weiss considers the facts that Cassio loves Desdemona and that Desdemona does not love Cassio. According to Hossack, the first fact combines the universal *love*, Cassio and Desdemona, in that order; the second fact combines the universal *negation, love*, Desdemona and Cassio, in that order. The combination relation treats all items of the vector except the first one as being on a par. The

first item relates all the others in the given order. Thus, in the first fact, *love* relates Cassio and Desdemona; and in the second *negation* relates *love*, Desdemona and Cassio. But that, Weiss argues, cannot be correct. Even though there is no fact in which *love* relates Desdemona and Cassio, *love* remains a relation even in the negative fact that Desdemona does not love Cassio. But Hossack must deny that there is anything relational about *love* in this fact. The best we might do, says Weiss, is to accept that negation is a further combination relation, rather than a universal, responsible for those facts that are negative. But this solves the problem at best for the case of *negation*, not for any other cases that might arise. And the problem is, Weiss argues, that there is an indefinite number of such other cases. In fact, the problem is one Russell had already faced with respect to propositions containing two verbs, amongst which are propositional attitude reports. There shouldn't be a special combination relation for each of those cases. Weiss concludes that due to the proliferation of combination relations, there is reason to doubt that we have a grasp of combination, and this threatens our understanding of what facts are supposed to be in Hossack's ontology.

Weiss's second argument concerns the evidence for knowledge being a relation between mind and fact. Even if one supports Hossack's anti-sceptical *motivation* which leads him to cut out epistemic intermediaries between the knower and that which is known, Weiss argues that the evidence Hossack produces for his way of doing so is weak. It is the factivity of knowledge that does the work here. True propositional attitudes are not factive, and so they are relations to contents, not facts. Knowledge works differently. And so the logic of belief attribution is very different from the logic of knowledge attribution, and the reason for this is that belief and knowledge are very different relations, and factivity can

be explained by the referentially transparent relation of mind to fact. Hossack's evidence for the claim that knowledge is a relation between mind and fact relies on the following linguistic argument. Knowledge-that claims entail knowledge-of claims: 'S knows that Fred is red' entails 'S knows of Fred's being red'. Sentences of the latter form require a sentential nominalization for their completion, and they are referentially transparent. S can only know of Fred's being red if Fred is red, and so here we have a direct relation between a mind and a fact. But there is no corresponding relation between a believes-that claim and a believes-of claim. Indeed, there are no believes-of claims at all.

Weiss isn't taken with this argument and observes that similar entailments hold for the propositional attitudes 'believes', 'hopes' and 'fears'. While these involve slightly different grammatical constructions, Weiss questions whether they are of logical significance. But as in these examples, the propositional attitudes must report a relation between a mind and a content, and so Hossack's linguistic argument fails to support his conclusion that knowledge is a relation between a mind and a fact.

Weiss's third argument is that Hossack's attempt to explain modality in terms of knowledge is implausible – a point on which he and **Dorothy Edgington** (Chapter 8) agree despite critiquing different versions of Hossack's argument for the same conclusion, Weiss the old version (Hossack 2007), and Edgington the new version published in the present volume.

According to Hossack's rationalist thesis, necessary facts coincide with those that are knowable *a priori*, that is, with those that have an *a priori* mode of presentation. Thus both notions, necessity and apriority, apply to facts. But, Weiss points out, apriority is usually taken to apply to contents, not facts. Hossack's fellow rationalists Leibniz, Kant and

Frege understood both notions in this way: for them, necessity and apriority apply to contents. Their coincidence faces the usual counterexamples. There are sentences that are known *a priori* but that are not necessarily true, such as those arising when a name is introduced by stipulation – e.g. Kripke’s famous example of ‘Neptune’ (Kripke 1980: 79, fn.33) – and there are sentences that are necessarily true, but that are known *a posteriori*, such as sentences expressing identities – e.g. ‘Hesperus is Phosphorus’ (Kripke 1980: 104). Hossack’s account may not succumb to those counterexamples, but Weiss discerns a change of subject. Whereas Hossack gives an account of *a priori* knowledge-of, we should expect an account of *a priori* knowledge-that. Weiss concludes that there is good reason to accept that apriority is a property of knowledge-that, and knowledge-that cannot explain necessity. Hence Hossack’s account fails.

Hossack doesn’t merely treat knowledge as unanalysable and metaphysically fundamental – in the sense problematised by Sainsbury. Part of putting ‘knowledge first’ means explaining other things in terms of it, that is, exhibiting its explanatory fruitfulness. (Hossack 2007: pp.xi–xvi) And, as we have just seen, part of the explanatory power of knowledge, for Hossack, is that it can be put to work in a reductive account of modality. This can be seen as the core tenet of his rationalism: Hossack’s *rationalist thesis* is that ‘a fact is necessary if and only if it has an *a priori* mode of presentation.’ (Hossack 2007: 125)

In ‘Necessity, Conditionals and Apriority’ (Chapter 7), **Keith Hossack** aims to prove that the propositions we can know by reason alone are necessary, and that the necessary propositions are those that are knowable *a priori* – in short, that the necessary and the *a priori* coincide. Given that necessity and possibility are interdefinable, Hossack takes himself to have ‘reduce[d] the modal to the epistemic’ (p.125).

Part of the interest of this position is that it violates a dogma enforced by followers of Kripke: that the *a priori* and the necessary do not coincide. This idea renewed interest in metaphysics, for it gave philosophy a new task: to reveal the broadly *a posteriori* necessities. This tied metaphysics closely to science in two ways. The results of science could amount to significant metaphysical (not merely physical) revelations, such as the identity of water and H₂O. And so too the methods of science could be harnessed for metaphysics, as Lewis infamously used inference to the best explanation to ‘discover’ the pluriverse. (Lewis 1986)

Hossack hopes, by contrast, to vindicate the traditional pre-Kripkean idea that the *a priori* and necessary coincide – but without going down the positivist route and identifying the necessary with the analytic. He takes the coincidence of necessity and apriority as falling out of Nelson Goodman’s definition of a counterfactual statement, according to which a counterfactual is true if its consequent is inferrible from its antecedent (Goodman 1979).

Hossack’s argumentative strategy is as follows:

1. Prove Goodman’s truth-conditions for counterfactuals.
2. Establish the definition of necessity in terms of inferrability.
3. Establish the definition of apriority in terms of inferrability.
4. Prove that Goodman’s truth-conditions for the counterfactual entail that necessity and apriority coincide.

This approach stands in stark contrast with David Lewis's, the best known even if not the most widely believed reduction of modality. Lewis wanted to reduce the totality of modal discourse to quantifications over the pluriverse, where the pluriverse is all the concrete possible worlds. (Lewis 1986, p. 7) So necessities become truths about what goes on in *all* such worlds, and possibilities become truths about what goes on in *some* of them. In Lewis's picture, there is no essential connection between apriority and truths about the pluriverse. Hossack, on the other hand, through the employment of the conceptually unanalysable and metaphysically fundamental notion of knowledge, sees an essential connection between mind and world, such that the necessities are available to knowledge.

Hossack's paper is put under the knife by **Dorothy Edgington** in her commentary on it (Chapter 8). She presents a cumulative argument whose conclusion is that 'I don't feel forced to the conclusion that the necessary and the *a priori* coincide' (p.141). Nor, by implication, does she think we should be either. She takes issue with the central notions of Hossack's attempted reduction of the modal to the epistemic.

Edgington takes the core notion of Hossack's account to be obscure. This is his notion of the *far-fetchedness* of a proposition which he employs instead of that of *closeness to actuality*. He uses *far-fetchedness* to 'complete' Goodman's definition of the counterfactual, such that it is non-circular and does not rely on an undefined or Lewisian notion of a possible world. *Far-fetchedness* is itself defined in terms of a counterfactual, thus making little advance on Goodman's own circular definition of *co-tenability*. Likewise, Edgington finds the notion of propositional *relevance* to be problematic.

Further, Edgington finds Hossack's proof of his version of Goodman's theory of counterfactuals to be convincing only for those who already accept that 'an ordering by

closeness/far-fetchedness is a key to the interpretation of counterfactuals.' (p.138) But she shows that such an ordering is not 'primitively evident' and that there is in fact substantial disagreement by competent philosophers on whether an ordering is 'a key to the interpretation of counterfactuals' and what the basis of the correct ordering might be.

Edgington is unconvinced by Hossack's final proof of the coincidence of necessity and apriority. According to Hossack, if P is necessary, then there is a proof of P. But as Edgington points out, as a consequence of Gödel's first incompleteness theorem, not every true proposition of number theory has a proof in the formal system of Peano Arithmetic. For instance, Goldbach's Conjecture is unproven. If it is true, it is necessarily true. It is counterfactually robust: whatever else were the case, it would be true. Thus, for Hossack, as it is *a priori*, it should be provable from self-evident premises. Many will agree with Hossack that an adequate notion of proof outstrips what is provable in the formal system of Peano Arithmetic: there are more ways of establishing propositions as true than there are ways of proving formulas in Peano Arithmetic. This is often taken to be a philosophical consequence of Gödel's Second Incompleteness Theorem and also of Tarski's Theorem about the undefinability of the truth predicate for arithmetic within Peano Arithmetic. But Hossack accepts that some proofs are infinitely large, such as proofs that appeal to the ω -rule. This rule requires infinitely many premises: it permits the derivation of the conclusion $\forall xFx$ from the infinitely many premises Fn , for every n . Goldbach's Conjecture may well have a proof in this sense (if it is true, calculate each instance and apply the ω -rule). But, Edgington points out, this goes against what most would ordinarily require of a proof, namely, that it enables us to come to know its conclusion on the basis of our knowledge of

the premises. This is not possible in a proof that appeals to the ω -rule, as we cannot come to know its infinitely many premises.

Edgington also provides counter-examples to the other direction of Hossack's equation of the necessary with the *a priori*, suggesting that there are *a priori* truths which are not necessary. For instance, 'What Dee weighs at t , is her actual weight at t '. So neither direction of the supposed coincidence are, for Edgington, plausible.

The thrust of Edgington's observations, counterexamples and arguments is that Hossack has not established what he had hoped: that the necessary and *a priori* coincide. And so the modal has not yet been reduced to the epistemic.

Questions relating to *a priori* knowledge and mathematical proof are further pursued by **Tamsin de Waal** in her paper 'The Mathematicians' Use of Diagrams in Plato' (Chapter 9). *A posteriori* knowledge is gained through experience. But how do we gain *a priori* knowledge? Mathematics is often cited as providing the paradigm case of an area of inquiry where *a priori* knowledge is to be found. Knowledge of mathematical propositions is usually acquired through proof. In geometry, these proofs often do, and in ancient Greek times essentially did, involve diagrams. But a diagram depicts a particular situation, while mathematical propositions are general. De Waal's central question is: how do we apprehend general mathematical propositions through diagrams in the proofs of geometry? How can a diagram that shows a particular triangle enable us to apprehend a general truth about all triangles? Geometry, as de Waal reminds us, was paradigmatic of all of mathematics in ancient times, and so her question was particularly pressing for Plato and his contemporaries. De Waal's paper concerns Plato's conception of mathematical knowledge, but her discussion draws general consequences for the epistemology of

mathematics, especially the cognitive significance of images and diagrams in gaining knowledge of mathematical propositions.

De Waal begins with a puzzle from Plato's *Republic*: while images are denigrated by Socrates and his friends, the way mathematics was done in Plato's time relied heavily on the use of images in diagrammatic proofs, and the study of mathematics plays a key role in the education of the philosopher-rulers. The denigration of images is a theme at various points of the *Republic*. In the image (!) of the Divided Line – Plato's representation of a hierarchy of four epistemic states at the end of book VI of the *Republic* (510dff) – images are relegated to the lowest epistemic state, that of *eikasia* or imagination. A common resolution of the puzzle is to take Plato as holding his contemporaries to account for using images in mathematical proofs, and finding mathematical practice defective where it does so. De Waal proposes an original alternative solution by arguing for the opposite: Plato endorses the use of images, where these are diagrams, as a fundamental and necessary feature of mathematics. They play an essential part in attaining the higher epistemic states of the Divided Line by facilitating the conversion from *pistis* to *dianoia* – translated by de Waal as *belief* and *understanding* – which is itself an essential step to reaching the goal of *noêsis* (rendered by de Waal as *intelligence/dialectic*).

De Waal's solution thus permits a full endorsement of Plato's comments that the study of mathematics *as it is* leads to 'a conversion and turning about of the soul' (*Republic* 521c) and will 'draw the soul away from the world of becoming to the world of being' (*Republic* 521d). This is possible despite the dubious role played by images in the *Republic*, because, argues de Waal, the diagrams of geometrical proofs prompt thought to direct itself towards the intelligible entities and force the recognition of the fact that

physical/visible objects are distinct from the intelligible entities and fall short of these in various ways. The positive role played by diagrams beyond the fact that they are a defining feature of the mathematical method lies in the way that the particular situation in a diagram represents a general truth of mathematics, and thus they point to the forms because they lead away from the particular to the general.

A feature that singles out mathematical propositions is their generality. This theme is picked up by **Nils Kürbis** in his paper 'Generality' (Chapter 10). Kürbis offers Hossack three arguments for his thesis that general facts are general in virtue of containing a special universal. To describe the world correctly, we need to use general statements. An important class of them contains those expressing the natural laws. As is the case with any other kind of statement, some general statements are true, some are false. In virtue of what are such statements true or false? Hossack rejects the account put forward by Wittgenstein in the *Tractatus*, according to which general statements are made true by their instances. (Hossack 2007: 68ff) This account was also the one held by Russell and Whitehead in the first edition of *Principia Mathematica*. (Russell and Whitehead 1910: 47) Hossack concludes that, instead, general statements are made true by general facts. These are facts that contain the universal *generality* in principal position. *Generality* is the referent of the universal quantifier of formal logic. Hossack does not, however, provide a sustained and detailed argument for the existence of general facts comparable to his argument for the existence of negative facts, those containing the universal *negation*. Kürbis's three arguments aim to fill this gap.

All three arguments are inspired by Russell's writings. The first two draw on Russell's analysis of the nature of universals as aspects of resemblance. All universally

instantiated universals resemble each other in this respect. Hence we should expect that there is a universal which is that resemblance. This is the universal *generality*. All existential facts resemble each other in this respect, so here, too, we should expect that there is a corresponding universal, namely *existence*. Together the universals *existence* and *negation* can account for generality; conversely, together the universals *generality* and *negation* can account for existence. Given the universal *negation*, we need not assume that *existence* and *generality* are both primitive, and Hossack decides in favour of *generality*.

The third argument follows a famous argument of Russell's that there must be universal facts, because any inference from purely particular premises to a general conclusion is invalid. Russell, however, was not explicit about the nature of general facts and cautious about what the right analysis of general facts might be. Kürbis argues that although Russell's argument leaves the nature of general facts open, the previous two arguments settle this question in Hossack's favour: we should draw Hossack's bold conclusion that what it is that makes a fact general is that it contains the universal *generality* in principal position.

Simon Hewitt's paper, 'We Belong Together? A Plea for Modesty in Modal Plural Logic', (Chapter 11) takes issue with one of Hossack's most important contributions in metaphysics and the philosophy of logic: atomism and plural reference. In his influential paper 'Plurals and Complexes' (2000), through the conceptual resources of plural logic, Hossack provides a reduction of our talk of complex objects to the pluralities of their parts. In effect, he argues for atomism, the view that there are no complex things, only simples, the atoms, and our ways of referring to the simples plurally.

Hewitt challenges an important thesis known as Plural Rigidity, according to which if a given thing is one of some given things, then it is necessarily so, given the existence of the relevant things. If Plural Rigidity is accepted, then Hossack's project of reducing our talk of complex objects to talk of pluralities of their parts seems to be undermined: by Plural Rigidity, complex objects will be necessarily identified with their actual parts. Thus, we will have a version of 'mereological essentialism'; the thesis that for any complex object x , if x has y as one of its parts, then y is a part of x in every possible world in which x exists. But mereological essentialism is often taken to be an implausible claim, for it suggests that to be part of something is to be part of its essence. It seems implausible to claim that given that my bicycle has a particular wheel as its part, then in every possible world in which my bicycle exists, it has that particular wheel as its part.

The Plural Rigidity thesis has received considerable attention in the recent literature. (See Linnebo (2016), Rumfitt (2005), Uzquiano (2011), and Williamson (2013)). The view has been challenged in Hewitt (2012). The rigidity of plurals has played an important role in a number of debates in metaphysics, philosophy of logic, philosophy of mathematics, and of language – for example, in the interpretation of higher-order logics; in Williamson's arguments for *necessitism*, the thesis that necessarily everything necessarily exists; and in Linnebo's (2010) programme for interpreting standard set-theory in a modal plural logic.

In his 'Plurals and Modals' (2016), Linnebo defends Plural Rigidity. He argues that Plural Rigidity lies at the core of a number of correct and plausible theses concerning the extensionality of pluralities. Linnebo starts from an analogue of the Necessity of Identity,

defended by Kripke (1980)⁶, for pluralities, to the effect that if every one of these objects is one of those objects and every one of those is one of these, then necessarily these just are those. In short: coextensive pluralities are necessarily coextensive:

$$(Cov) \quad xx \equiv yy \rightarrow \Box (xx \equiv yy)$$

where ' $xx \equiv yy$ ' abbreviates ' $\forall u(u < xx \leftrightarrow u < yy)$ ', stating that whatever is part of the plurality xx is part of the plurality yy , and *vice versa*. ('<' stands for the parthood relation.) As Linnebo puts it, (Cov) tells us that 'two overlapping pluralities necessarily covary in their membership' (2016, p. 662). But it is not difficult to see that (Cov) does not entail the desired Plural Rigidity, which states that a plurality has the same members at any world in which it exists. Linnebo's strategy for deriving Plural Rigidity is through some further assumptions, to bridge the gap between the extensionality of pluralities and their rigidity.

Hewitt's contribution takes issue with Linnebo's formal arguments in favour of Plural Rigidity. In Hewitt's view, all of Linnebo's arguments suffer from the same type of problem: they fail to persuade somebody not already committed to Plural Rigidity. In fact, the source of the difficulty, in Hewitt's view, lies in Linnebo's starting point: the extensionality of pluralities as codified by (Cov). It is exactly this extensional conception of pluralities that is question-begging.

⁶ The first published proof that, under certain assumptions, identity statements are necessary is by Ruth Barcan Marcus (1947). The simple proof that this follows by replacement of the Law of Identity in the schematic first-order version of Leibniz' Law was apparently first published by Quine (1953). See Burgess (2014). The simple proof is also found in Prior (1955: 205f). Prior notes that the simple proof follows from a more complex one, essentially due to Barcan Marcus, if the Law of Identity and the first-order version of Leibniz' Law are taken as axioms rather than derived from Leibniz' second-order definition of identity.

One of the auxiliary assumptions that Linnebo employs to derive Plural Rigidity is Uniform Adjudication, which states that necessarily, to be one of these things and that thing is to be one of these things or to be identical with that thing. The proof of Plural Rigidity rests on (Cov) and Uniform Adjudication. But Hewitt argues that the use of (Cov) in Uniform Adjudication begs the question against someone not antecedently committed to Plural Rigidity. Hewitt makes a similar point concerning the other two arguments discussed by Linnebo.

Hewitt points out that there is also linguistic evidence against Plural Rigidity, but he concedes that such evidence cannot provide us with an effective tool against the use of the principle in our theorizing about the plural concepts we express in our natural languages. In Hewitt's view, in the absence of a philosophical motivation for Plural Rigidity, the appeal to it in support of strong metaphysical theses is illegitimate.

The contributions of **Øystein Linnebo**, **Bahram Assadian**, and **Peter Simons** deal with questions in the philosophy of mathematics, which have direct interactions with Hossack's concerns. As discussed above, according to Hossack, numbers are properties. The natural numbers are properties of pluralities, the positive real numbers are properties of continua, and the ordinal numbers are properties of series. Central to Hossack's project is the Magnitude Thesis, which, to put it roughly, is the thesis that a magnitude is a property that is shared by equivalent quantities. In his contribution, 'Aristotelian Aspirations, Fregean Fears: Hossack on Numbers as Magnitudes', (Chapter 12), **Øystein Linnebo** draws our attention to a structural similarity between the Magnitude Thesis and Fregean abstraction principles. He then explores the question as to how different Hossack's

magnitudes really are from mathematical objects, as conceived of by Frege and the neo-Fregeans.

Let us say that two pluralities are equivalent just in case the equivalence relation of one-one correspondence holds between them. Thus construed, the Magnitude Thesis involves a function that maps a quantity to its magnitude. Fregean abstraction principles behave in a similar way. Consider Hume's Principle, which states that the number of the concept F (i.e. the number of things falling under the concept F) is the same as the number of the concept G just in case there is a one-one correspondence between the F s and the G s. The cardinality function, denoted by 'the number of', maps a concept to its cardinal number. So, we can put forward a Frege-style abstraction principle for magnitudes as follows:

$$(AP) \quad \phi(x) = \phi(y) \leftrightarrow x \sim y$$

where ' $\phi(x)$ ' stands for the function *the magnitude of*. Thus, (AP) states that the magnitude of the quantity x is identical to the magnitude of the quantity y just in case x and y are equivalent, for some suitably chosen equivalence relation, depending on the sort of quantities x and y are.

In Hossack's view, there is an important reason for favouring his Aristotelian approach concerning the nature of numbers over the Fregean one. The entities on the left-hand side of (AP), the magnitudes of certain quantities, belong to the category of properties, not Fregean objects. And since magnitudes are sorts of entities that could be acceptable for nominalists who deny abstract *objects*, the Aristotelian approach is preferable.

Linnebo, however, points out that Hossack's axiomatization of the mereological structure of quantities, as developed in Hossack (2020, ch 4), has been formulated in a type-free language, in which a single type of variable is used to range over individuals, pluralities, continua, and series. And this, Linnebo claims, 'gives us all that Frege and his followers ever wanted when they defended the idea of numbers as objects.' (p.197) All that Frege's famous bootstrapping argument for the existence of infinitely many natural numbers needed is that numbers figure as members of the pluralities whose numbers we are interested in. This condition, however, does not require a substantial metaphysical notion of objecthood. Thus, if Hossack's conception of numbers is distinctively non-Fregean, his numbers must fail to be objects in some metaphysical sense, but it is not clear whether Frege and his neo-Fregean followers need such a metaphysically loaded conception of the objectual nature of numbers.

Linnebo further argues that the abstractionist reading of the Magnitude Thesis in terms of (AP) exposes (AP) to a version of what is known in the literature as the *bad company problem*. According to this objection, abstraction principles cannot be acceptable ways to introduce mathematical concepts, because there is no satisfactory way to separate 'good' abstraction principles, such as Hume's Principle, from 'bad' ones, such as Frege's ill-fated Basic Law V, with which they share their general form. What is the principled account that separates the good from the bad abstraction principles? And how can Hossack respond to the bad company problem?

This question is pressing in view of Hossack's commitment to Unrestricted Comprehension for Series; i.e. the principle that if a two-place formula defines a well-ordering without repetitions, then the formula defines a series. The threat of the Burali-

Forti Paradox is familiar here: by Unrestricted Comprehension for Series, there would be a series of all the ordinals, and by Hossack's theory, this series would define an ordinal which would be greater than all the ordinals – including itself. This paradox forms a special instance of the bad company problem.

Hossack's proposal (2020, ch. 10) for blocking the Paradox is to restrict the range of series only to those that are defined by *recursive* well-orderings. Linnebo, however, argues that Hossack's proposal is 'both unacceptably restrictive and ultimately *ad hoc*.' (p.199) It is restrictive because, for example, the uncountable plurality of the real numbers can be well-ordered, but the condition of recursivity prevents us from making serial reference to the real numbers in that order. And it is *ad hoc*, because there is no connection between Hossack's responses to the Burali-Forti Paradox and Russell's Paradox: in the latter case, Hossack makes a Quinean move by appealing to NFU; i.e a version of Quine's New Foundations with entities which are not sets, known as *Urelemente*. But in the former, Hossack restricts the range of series to recursive well-orderings. The different explanations for similar set-theoretic paradoxes need a strong justification, which, in Linnebo's view, is lacking in Hossack's proposal.

The main focus of **Bahram Assadian's** paper 'Mathematical Structures, Universals, and Singular Terms' (Chapter 13) is structuralism in the philosophy of mathematics; the thesis that foundational mathematical theories are about *abstract structures*. But what is an abstract structure? This question is related to Hossack's detailed account of the nature of ordinal numbers (2020, ch. 10), according to which an ordinal number is a universal: it is what isomorphic serial relations, i.e. isomorphic well-ordered series, "have in common", where two serial relations R and S are isomorphic if and only if there is an order-preserving

bijection f from the field of one onto that of the other; i.e. the function f is such that: $\forall x \forall y (Rxy \leftrightarrow Sfxfy)$. An ordinal, thus construed, is a particular instance of an abstract structure, which is what isomorphic relations, and not necessarily well-ordering relations, instantiate or have in common. Assadian's paper can thus be seen as a way of extending Hossack's conception of ordinals to an account of structuralism in which abstract structures are identified with universals.

In the literature on structuralism, abstract structures are what isomorphic 'models' or 'systems' of a mathematical theory have in common, where a system, following Shapiro (1997), is defined as an ordered $(n+1)$ -tuple consisting of a domain D and relations R_1, \dots, R_n on this domain. A much-discussed account of abstract structures, known as *ante-rem structuralism*, is defended and developed by Shapiro (1997, 2008). According to this view, an abstract structure corresponding to a mathematical theory is a *structural universal* shared by the isomorphic systems of theory, such that the universal consists of a category of purely structural entities, known as *positions*. The question Assadian deals with is this: could an *ante-rem* structure be the kind of entity the structuralist has been looking for as a candidate for *abstract* structures, namely as a candidate for what isomorphic systems of a theory have in common? Assadian discusses two arguments leading to a negative answer and offers what the structuralist needs in order to resist these arguments.

The first argument is Geoffrey Hellman's (2001: 195-6) permutation objection. Hellman argues that *ante-rem* structures themselves, just like their instantiating systems, are subject to the sort of indeterminacy Benacerraf (1965) has argued for. Thus they leave us with distinct and yet isomorphic copies. The reference to positions of structures will be

as indeterminate as reference to the elements of their corresponding systems. Referential indeterminacy does not go away despite postulating *ante-rem* structures.

Assadian argues that the source of the problem of referential indeterminacy lies in the structuralist's conception of positions as *bona fide* objects, as the referents of semantically singular terms. He thus puts forward a thesis, recently defended by Hale and Wright (2012), Hale and Linnebo (2020), and Hossack (2020, ch.1 and ch.4), according to which singular terms can refer to universals. Thus, contrary to what Frege recommended, it is not the case that *only* objects can be the referents of singular terms. This will liberate the structuralist from positing a special *system* with a domain of positions-as-objects to serve as the unique referents of our mathematical expressions: we can successfully refer to the universal *natural number structure* by using the singular term 'the structure of natural numbers', as opposed to the systems having this universal in common. The universal *natural number structure* is a structural universal, and we can permute its sub-universals, i.e. its positions, in a way that produces an isomorphic system of that structure. This isomorphic system itself is not the universal *natural number structure*. All the same, we can still use the singular term 'the structure of natural numbers' to refer to the universal rather than to a system obtained from it by permuting its sub-universals. The idea that the singular terms formed by an abstraction principle can be understood as "derived" reference to universals comes up also in Linnebo's paper in this volume, where he provides a reconciliation between Hossack's Aristotelianism on the nature of numbers as universals and Frege's abstractionism.

The second argument against the *ante-rem* conception of abstract structures is due to John Burgess (1999). According to this argument, an *ante-rem* structure is supposed to

be a system, which is an ordered set, and so it is supposed to consist of a collection of positions-as-objects, with distinguished relations on them. But abstract structures as what isomorphic systems have in common – what Burgess calls ‘isomorphism types’ – are abstracts with respect to the equivalence relation of isomorphism. Since the *criterion of identity* for abstract structures and that of *ante-rem* structures are distinct, the two categories cannot overlap. Assadian argues that Burgess’ argument rests on the conception of *ante-rem* structures as special systems whose positions are taken to be objects. Again, once we dispense with this objectual perspective, there will be room for the identification of *ante-rem* structures, taken as structural universals, with abstract structures.

Peter Simons’ paper, ‘Arithmetic in a Finite World’ (Chapter 14), defends a nominalist account of numbers, and thus sharply contrasts with Hossack’s realism in the philosophy of mathematics. In fact, Simons’ starting point accords well with Hossack’s thesis that numbers are properties – for example, the natural numbers are properties of collections. But Simons defends a radically contrasting thesis concerning these properties. In his version of nominalism, these properties are *fictions*. In particular, Simons offers a nominalistic reading of Hume’s Principle. In his view, numerical expressions of the form ‘The number of *Fs*’ are not genuinely referential expressions.

Starting with the natural numbers, Hossack’s thesis faces the familiar problem for accounts which take mathematical statements to be grounded in individuals, pluralities, or collections of them: how can it be ensured that there are enough individuals, pluralities, or collections to validate the axioms of Peano Arithmetic?

Simons aims to ground arithmetical truths, which require infinitely many objects, in a finite (concrete) world. Frege’s proof of the infinity of the natural numbers, also discussed

in Linnebo's contribution, indispensably depends upon the objectual nature of numbers – at least upon numbers' being objects in a general logical sense. But in a Russell-style type theory, the Fregean proof is not available. For Russell, a number is a class of classes, and thus belongs to at least level 3 in the hierarchy, two levels up from level-0 individuals. As a result, Russell has to appeal to the Axiom of Infinity, which asserts that there are infinitely many individuals. However, as Russell was aware, and Simons reminds us again, the logical character of the Axiom of Infinity is highly dubious. Russell writes:

Nor will it do to add the *Axiom of Infinity* as a hypothesis to theorems in whose proof it is used – a procedure which is adopted in *Principia Mathematica*. To have recourse to this tactic is to abandon the logicist project – supposing we had a satisfactory axiomatization of a physical theory, no one would be taken seriously who claimed to reduce that branch of physics to logic by rewriting each theorem of the theory as a conditional with the conjunction of the axioms as antecedent. When the number theorist asserts that there are infinitely many primes, he is making a categorical assertion, and not merely claiming that if there are infinitely many individuals, then there are infinitely many primes. (Russell, 1919c, p.141; see also pp. 202–3)

Simons reminds us that the main reason for Whitehead and Russell, in their *Principia Mathematica*, to postulate the non-logical Axiom of Infinity was to guarantee the existence of an infinite number of objects. Non-logicality aside, Simons also holds that the Axiom of Infinity has the 'disturbing consequence that Peano arithmetic may be empirically false, if the world is finite in individuals.' (p.222)

Given the assumption of the existence of at least two individuals, and also the existence of higher-order collections (pluralities) – collections of collections, collections of collections of collections, and so on – Simons shows, in a type-style hierarchy, how we can capture the axioms of the Dedekind-Peano Arithmetic; hence the title of his paper, ‘Arithmetic in a Finite World’. Collections of individuals are different from the individuals of the collections. The collection of the Moon and the Sun is not identical to the Moon and the Sun. A collection is not an individual, for otherwise there would be infinitely many individuals. But the mere existence of the Moon and the Sun ensures the existence of their collection. So, we have at least three items if we have the Moon and the Sun: the Moon, the Sun, and their collection. But we will also have a higher-order collection: the collection of these three items. Each level, based on two or any finite number of individuals, is finite, but since there is no upper limit to the orders, infinitely many collections appear in the hierarchy.

An interesting metaphysical consequence of Simons’ account is that ‘Peano arithmetic is not necessarily true’ (p.222), for its truth requires as an additional premise that there are at least two individuals.

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