Harmony, Normality and Stability

Contents

1 Conceptual Considerations on Harmony

1.1 Gentzen's Observation and Gentzen's Thesis

Gentzen observed a 'remarkable systematic' in the 'inference patterns' for symbols of the calculus of natural deduction and suggested that 'by making these thoughts more precise it should be possible to establish on the basis of certain requirements that the elimination rules are functions of the corresponding introduction rules.'[1] One of the objectives of this paper is to fill this specify such a function: I will specify a process by which it is possible to determine the elimination rules of logical constants from their introduction rules, and conversely, to determine the introduction rules from the elimination rules.

I will then use this result to clarify some issues surrounding a famous remark of Gentzen's. The observation of the 'remarkable systematic' lead Gentzen to put forward what might be called 'Gentzen's Thesis': 'The introductions constitute, so to speak, the "definitions" of the symbols concerned, and the eliminations are in the end only consequences thereof, which could be expressed thus: In the elimination of a symbol, the formula in question, whose outer symbol it concerns, may only "be used as that which it means on the basis of the introduction of this symbol".'[2] Gentzen's Thesis invites being fleshed out in a comprehensive theory, which is of course what Michael Dummett has done in his *proof-theoretic justification of deduction* or *proof-theoretic semantics.* Dummett employs the notions of *harmony* and *stability* to specify which rules of inferences can count as defining the meanings of the logical constants they govern. The intuitive philosophical content of harmony and stability is that harmony obtains if the grounds for asserting a proposition match the consequences of accepting it, and stability obtains if the converse also holds. Rules of inference define the meanings of a logical

[1] Gerhard Gentzen: 'Untersuchungen über das logische Schließen', *Mathematische Zeitschrift* 39 (1935), 176-210, 405-431, p.189

[2] *Ibid.*

constant they govern if and only if they are stable. There are, however, *two* notions of harmony at play in Dummett's work. One of them has a formally precise characterisation in terms of Gentzen's cut-elimination as transposed to natural deduction, i.e. in Prawitz work on the normalisation of deductions. A deduction in normal form can be described, in Gentzen's words, as on without detours: it is a particularly direct deduction[3]. This is a result applying to a logic in which rules of inference occur. The other notion of harmony is more difficult to pin down. Dummett seems to intend this notion of harmony to apply to the forms of rules of inferences no matter what logic they might occur in. In Dummett's writings, this notion is never made formally precise, but, as I shall argue, Gentzen's functions can be used to achieve this aim. Stability, too, is not as clear as normalisation, but I shall argue that Gentzen's functions provide us with a way of achieving a formally precise characterisation of this notion, too.

The formal details are given in section 2. The next section contains conceptual considerations about what harmony and stability amount to. Although I'll quote quite extensively from Dummett's *The Logical Basis of Metaphysics* (henceforth *LBM*), my aim is not exegetical. My aim is to provide a formally precise way of defining harmony and stability on the basis of Dummett's work. This is will capture much of what Dummett intends these notions to convey, but it is not exactly what he had in mind, because on my account, classical negation as well as intuitionist negation turn out to be governed by harmonious rules, whereas Dummett thinks this is only holds for the latter.

I end the paper with a discussion of a conjecture of Dummett's concerning the relation between harmony, stability and conservative extensions.

[3] *Ibid.*, p.177

1.2 Dummett on Harmony, Normalisation, Conservative Extensions and Stability

Dummett singles out two features of the use of expressions that are of central importance for specifying their meanings. The two features are intended to apply very generally to all kinds of expressions, but I'm only concerned with the logical constants. 'The first category [of principles governing our linguistic practice] consists of those that have to do with the circumstances that warrant an assertion [...] we need to know when we are *entitled* to make any given assertion, and when we are *required* to acknowledge it as true. [... Furthermore,] in acquiring language, we learn a variety of principles determining the consequences of possible utterances; these compose the second of our two categories of principles that govern our linguistic practices.' (*LBM* 211f) Applied to the logical constants, the first feature of their use corresponds to applications of introduction rules, the second one to applications of elimination rules in a calculus of natural deduction.

Dummett's informal explanation of harmony is that it is a relation that ought to hold between these two features of the use of expressions. 'The two complementary features of any practice ought to be in harmony with each other [...] The notion of harmony is difficult to make precise but intuitively compelling: it is obviously not possible for the two features of the use of any expression to be determined quite independently. Given what is conventionally accepted as serving to establish the truth of a given statement, the consequences cannot be fixed arbitrarily; conversely, given what accepting a statement as true is taken to involve, it cannot be arbitrarily determined what is to count as establishing it as true.' (*LBM* 215) Thus, in the case of the logical constants, the grounds for asserting a formula with main operator Ξ, i.e. the conditions under which an introduction rule for Ξ can be applied, should match, in some way to be made precise, the consequences of asserting a formula with main operator Ξ, i.e. the conditions under which an elimination rule for Ξ can be applied. The converse should also hold, which will be of importance for the notion of stability. Thus the introduction rules should somehow determine the elimination rules, and conversely, the elimination rules should somehow determine the introduction rules for Ξ.

When Dummett applies the notion of harmony to the logical constants, he gives two quite different ways of spelling it out formally. One is connected to the notion of a *conservative extension*. Let L_1 be a logic with language $\mathcal{L}_1$, a deductive system $\mathfrak{R}_1$ and a consequence relation $\vdash_{\mathrm{L}_1}$; let L_2 be a logic with language $\mathcal{L}_2$ and a deductive system $\mathfrak{R}_2$ extending $\mathcal{L}_1$ by new symbols and $\mathfrak{R}_1$ by rules for them, resulting in a consequence relation $\vdash_{\mathrm{L}_2}$. Then $\vdash_{\mathrm{L}_2}$ is a *conservative extension* of $\vdash_{\mathrm{L}_1}$ iff, $X \vdash_{\mathrm{L}_2} B$ iff $X \vdash_{\mathrm{L}_1} B$, if $X, B \in \mathcal{L}_1$. 'What is it for the introduction rules and the elimination rules governing a logical constant to be in harmony? We saw that harmony, in the general sense, obtains between the verification-conditions or application-conditions of a given expression and the consequences of applying it when we cannot, by appealing to its conventionally accepted application conditions and the invoking the conventional consequences of applying it, establish as true some statement which we should have had no other means of establishing: in other words, when the language is, in a transferred sense, a conservative extension of what remains of it when the given expression is deleted from its vocabulary.' (*LBM* 247) This is one notion of harmony.[4]

Following this (*LBM* 247ff), Dummett characterises another notion of harmony, connected to normalisation of deductions. Dummett demands that for harmony to obtain between introduction and elimination rules for a logical constant Ξ *maximal formulas* with Ξ as main connective can be removed from deductions, where a maximal formula is one that has been

[4]It is worth noting a difficulty with Dummett's discussion: failure of conservativeness gives us no reason to blame a constant, rather than some feature of the fragment of the language without it.

introduced by an application of an introduction rule and is major premise of an application of an elimination rule. Dummett calls the context in which a maximal formula occurs in a deduction a *local peak*, and thus harmony obtains if local peaks can be removed from deductions. Thus the requirement of harmony between introduction and elimination rules for a logical constant Ξ gets its formal content from the requirement that any maximal formula may be removed from deductions and local peaks may be levelled by applying reduction procedures. Reduction procedures are methods for reordering deductions in such a way that these 'detours' are avoided. Applying the reduction procedures should always turn deductions into new deductions. Trivial as this observation may sound, it shows that the *point* of having harmonious rules is not that they exhibit some feature independently of any logic in which they might occur, but rather that harmony, so understood, is a feature *relative to the logic* a rule is part of.

Normalisation is quite different from conservativeness. Depending on how classical logic, for instance, is formulated, its negation may be conservative over its positive fragment, although deductions don't normalise. I will, however, show later that, under certain conditions, normalisability entails conservativeness.

To keep the two notions of harmony apart, Dummett concludes that 'we ought, therefore, to distinguish between "intrinsic harmony" and "harmony in context", or "total harmony". We may continue to treat the eliminability of local peaks as a criterion for intrinsic harmony; this is a property solely of the rules governing the logical constant in question. For total harmony, however, we shall demand that the addition of that logical constant produce a conservative extension of the logical theory to which it is added. This notion is in a high degree relative to context, that is, the base theory to which the addition is being made.' (*LBM* 250) In the following, I won't call 'total harmony' harmony at all, but stick to conservativeness. It is the notion of intrinsic harmony I am mostly interested in.

There is a certain ambiguity in the notion of intrinsic harmony. On the one hand, as the quote in the last paragraph shows, Dummett presents it as a feature that applies to rules of inference *independently* of formal systems they form part of. At the same time, the criterion Dummett suggests for whether intrinsic harmony obtains is one that can only be applied to formal systems that rules are part of, not to rules in isolation: whether local peaks can be levelled depends on whether the reduction procedures for removing local peaks from deductions always turn deductions into deductions, which is a feature that is applicable only to rules as part of a formal system. Dummett's initial discussion of harmony also suggests a notion of harmony that applies to the form of rules of inference: introduction rules should determine the elimination rules, as the grounds for asserting a proposition should match the consequences of accepting it. We might say that looking at the introduction rule alone, we should be able somehow to 'read off' its elimination rule. As such, as a notion applying to the forms of rules of inference, this notion is independent of formal systems, even though, of course, the *point* of having a rule of inference is to have it as part of a formal system.

That Dummett envisages a notion of harmony that applies to the shape of rules of inference independently of formal systems is corroborated by his discussion of stability. Stability is a stronger notion than harmony. 'If there is harmony between these conventional consequences and the grounds we admit for asserting [a statement], this guarantees that we shall not assert it when its meaning does not justify our doing so, that we do not treat as a ground for it what should not warrant the consequences that we draw. It does not show that we should be willing to assert the statement whenever those consequences would be warranted, and hence whenever we should be entitled to do so. [. . .] The demand that such a condition be met goes beyond the requirement of harmony: we may call it stability.' (*LBM* 287) Dummett describes a process of justifying introduction rules relative to elimination rules and conversely, which is a process in which formal systems don't come in, at least initially. Looking at the introduction rules alone we should be able to determine which elimination rules are harmonious with them, and conversely. 'If we use an upwards justification procedure, harmony validates a putative elimination rule; if we use a downwards justification procedures, it validates a putative introduction rule. In either case, harmony is guaranteed between valid rules. But, to verify that stability obtains, we have to appeal to *both* justification procedures. Suppose that we adopt the downwards justification procedure, and start with a set $\mathcal{E}$ of elimination rules. By our procedure, we can determine which introduction rules are valid: say these form a set $\mathcal{I}$. Now, with respect to this set of $\mathcal{I}$ of introduction rules, the upwards justification procedure is well-defined: so we can use it to determine which elimination rules are valid, according to the criteria of the upwards procedure. If we get back by this means to the set $\mathcal{E}$, or some set interderivable with $\mathcal{E}$ in the ordinary sense, in the presence of $\mathcal{I}$, stability prevails. Otherwise not. [...] Obviously, if we had adopted the upwards justification procedure, with a set $\mathcal{I}$ of introduction rule as basis, we could perform the converse test for stability. First finding the set $\mathcal{E}$ of elimination rules, we could apply the downwards procedure to discover which introduction rules were validated by it. If we got back to the set $\mathcal{I}$, stability would obtain; otherwise not.' (*LBM* 288) In this section, Dummett only talks of rules of inference which are determined from each other, so the criterion is one applied to the forms of rules of inference by themselves, not relative to formal systems they may be part of. Dummett, it is fair to say, does not in fact specify a general procedure for determining introduction and elimination rules from each other.[5] The function Gentzen speaks of,

[5]He does, however, specify ways of deciding whether a give set of introduction and elimination rules satisfies criteria of validity: but these don't actually allow us to determine

which I will give in section 2.2, will do precisely that.

Finally, Dummett's Conjecture concerns the relation between the two notions of harmony. 'Although this distinction [between total and intrinsic harmony] was drawn in the preceding chapter, we have since proceeded as though intrinsic harmony was all that mattered; but it is total harmony that must prevail if the *point* of the requirement of harmony is to be attained, namely that, for every logical constant, its addition to the fragment of the language containing only the other logical constants should produce a conservative extension of that fragment. We may conjecture that the problem is a minor one, however: that is, that intrinsic harmony implies total harmony in a context where stability prevails.' (*LBM* 290) This conjecture stands in need of interpretation, as I'll show in the last section.

1.3 Towards Definitions of Harmony and Stability

1.3.1 Two Notions of Harmony

The discussion of the last section suggests that besides Dummett's notion of total harmony or conservativeness, here are two furhter notions of harmony at play in Dummett's writings:

1.) Harmony is a feature detectable in the rules governing a logical constant: Introduction and elimination rules for a constant are *in harmony* if the latter can somehow be *read off* the former (or conversely): the form of the introduction rules determines the form of the elimination rules (or conversely). In the following I will sometimes call this notion of harmony 'the first notion of harmony', or simply 'harmony'.

2.) Harmony holds if the rules for a constant yield a suitable induction clause for a *normalisation theorem*: Introduction and elimination rules for a constant are *in harmony* if *maximal formulas* may be removed from deductions. To distinguish this notion from the first, let's call it *normality*, but

rules of inference, so I won't go into that here.

I'll sometimes refer to it as 'the second notion of harmony'.

The first notion of harmony is a feature of rules *independently* of a logic they are part of. It is a common feature all rules of a certain kind are supposed to exhibit. It is to do with a special *uniformity in the form* of these rules—the focus here is on Gentzen's 'remarkable systematic'. Contrary to that, normality is a feature rules can only have *relative to a logic* they are part of—the focus here is on Gentzen's notion of removing detour from deductions. There is no suggestion that there is a common feature of all rules which occur in logics which normalise.

As suggested by Dummett's writings, the relation between the two notions of harmony is the following. It has not been made formally precise what how to determine elimination from introduction rules or conversely. Dummett fails to specify a method for doing so. Thus the first notion of harmony remains an informal notion, which has no precise formal content that would allow us to establish whether rules of formal systems are in harmony. Contrary to that, normality is a formally precise notion. Thus normality suggests itself as giving formally precise content to the notion of harmony.[6]

The path Dummett choses here, however, creates a certain tension. There is a tendency in Dummett's writings to make harmony a notion as independent of specific formal systems as possible. At the same time the way the notion is made precise ties it rather closely to specific formal systems—indeed, it is nonsensical to sever normality from the context of a specific formal system. But it is crucial to realise what the point of harmony is: namely that harmonious rules lend themselves in a particularly general and straightforward way as the basis of normalisation proofs, as I shall show in

[6]Notice also, incidentally, how this dialectics of the development of the two notions of harmony manifests itself in Chapter II of Prawitz' *Natural Deduction*, which is entitled 'The Inversion Principle' and starts of with a discussion of harmony and ends with the specification of the *reduction procedures* for removing maximal formulas from deductions, *i.e.* the crucial machinery needed for establishing normality of rules of inference of a logic.

section 2: the point of the first notion of harmony is to enable us in a general way to specify rules of inference for formal systems in which deductions normalise.[7]

What is more problematic is the following. If Dummett suggests that normality provides a formally precise way of cashing out the notion of harmony, then how are we to cash out formally the notion of stability? His way of spelling out Gentzen's Thesis is to lay down that the meaning of a logical constant may be defined by introduction and elimination rules if they are *stable*. Thus it is the notion of stability which is crucial to Dummett's proof-theoretic semantics. Stability is supposed to be harmony plus some kind of converse of harmony. But normalisation is only a formal criterion for harmony, not stability: it has no suitable converse. So we have no formally precise way of cashing out the crucial notion of stability. And that's a problem, because in the absence of a formally precise notion of stability, it is hard to see what role it could play in proof-theory.

Dummett does, however, hint at a way of cashing out the notion of stability, when he talks about determining elimination rules from introduction rules and conversely. In section 2.2 I will make this idea precise by explicating Gentzen's 'remarkable systematic' between the forms of introduction and elimination rules, and use this to define a notion of harmony in section 2.3. However, it'll turn out that the notion of stability which arises from demanding the reversability of the process of 'reading off' introduction rules from elimination rules or elimination rules from introduction rules would not define a notion of stability which adds anything to harmony: they would be equivalent. To extract a substantial notion of stability, I suggest we look at some other things Dummett says about this notion. But first, I'll say a bit

[7] As a matter of fact, of course, in, for instance, intuitionist logic, the harmony the rules for its constants exhibit suffices for their normality to hold as well. But this is due only to the special features of intuitionist logic, not *per se* to the notion of harmony. It will emerge in the next section that there are rules which Dummett thinks exhibit harmony$_1$, but they can occur in a logic where normality fails.

more about the notion of harmony I am going to define and its relation to normalisability.

1.3.2 Harmony

As already mentioned, the idea for my answer to the question of how to make the first notion of harmony formally precise can be found in Gentzen's observation of the 'remarkable systematic' in the inference patterns and that it should be possible to specify a function mapping introduction rules onto elimination rules. Gentzen does not say anything more about the function, and as far as I know there is no attempt in the literature to specify it.[8] Dummett tries to capture some of Gentzen's idea with his notions of harmony and stability, but what he has to say on that issue is a long way from the envisaged mathematical precision of Gentzen's remark. Specifying Gentzen's function gives a formally precise content to the first notion of harmony: the mapping of introduction to elimination rules the function provides explains what it means to determine or 'read off' elimination rules from introduction rules. The process I am going to specify can be inverted, so that elimination rules can also be mapped onto introduction rules, but this does not result in a suitable definition of the notion of stability: it is trivially the case that the process can be inverted, hence if stability is merely 'harmony plus its converse' it would not have any content over and above that of harmony.

Previous discussions have, to my knowledge, assumed that there is *one* kind of rule of inference and that the meanings of the connectives are given either uniformly by introduction rules *or* by elimination rules, the others being determined relative to them by some principle of harmony. In section 2.2 I shall give *two* kinds of general forms of rules of inference, one where initially an introduction rule for a connective is given and one where initially an

[8]Zucker and Tragesser do not quite succeed in doing so, as on their account there are occasions where a set of introduction rules is given, but 'there does not seem to be a suitable set of [elimination] rules' ('The Adequacy Problem for Inferential Logic', p.506).

elimination rule is given, and I shall specify for each kind a general method for determining the elimination and introduction rules for the connective.

I shall, however, loosen the ties between harmony and stability on the one hand and normalisation on the other. I shall define harmony and stability to be features of operational rules that are indeed independent of any formal systems they may occur in: they are features operational rules exhibit merely by virtue of their form. Contrary to that, normalisation is to do with deductions in formal systems. The point of having these notions of harmony and stability is of course that they nonetheless play a vital role in the proof of normalisation theorems. The philosophical adequacy of the notions of harmony and stability to be defined is ensured by the fact that they give rise to reduction procedures for eliminating detours from deductions. In some logics, these reduction procedures may not guarantee normalisability, because they may fail to turn deductions into deductions. To show that the deductions of a logic do normalise, i.e. to establish normality, it needs to be established that the reduction procedures have this property when applied to deductions of that logic. In a large class of cases, this is particularly straightforward, namely for logics containing only stable rules, on my definition of stability, because the form of those rules guarantees normalisability.

1.3.3 Normality

The formal framework to be set up in detail in section 2.1 is a generalisation of the Gentzen calculi used in substructural logics, where operational rules for the logical constants as well as structural rules for the reordering of assumptions are applied to expressions of the form $X \vdash B$ and the initial expressions are of the form $A \vdash A$.

The discussion so far has focussed on the removal of maximal formulas form deductions. Normalisation also needs to take into account another phenomenon. Consider a rule like $\vee$E:

$$\frac{X \vdash A \vee B \qquad Y(A) \vdash C \qquad Y(B) \vdash C}{Y(X) \vdash C}$$

Suppose the minor premise C of an application of $\vee$E has been introduced by an application of an introduction rule for its main operator and the conclusion of the application of $\vee$E, of the same shape as C, is major premise of an elimination rule. These formulas C are just as problematic as a maximal formula: whatever philosophical issues are connected to the latter are connected to the former. It needs to be shown that applying a rule like $\vee$E does not disturb the equilibrium of grounds for asserting C and consequences drawn from it. In other words, these occurrences of C should be removable from deductions.

In fact, one should expect something slightly stronger to be required. A rule like $\vee$E introduces new grounds for asserting C. Hence we should have to show that these new grounds balance exactly the consequences of accepting C. Thus we should show that a formula C derived by an application of a rule like $\vee$E and major premises of an elimination rule can be removed from deductions.[9] As far as I know, it is normally only stipulated that a formula introduced by *ex falso quodlibet* which is also major premise of an elimination rule counts as a maximal formula should therefore be removable from deductions. This is in fact a special case of the previous one, as will become clear later. I'll prove the more general case in section 2.5.2.

Similarly, a formula introduced by *consequentia mirabilis*, which allows the derivation of a formal if its negation leads to absurdity, and major premise of an elimination rule counts as a maximal formula which should be removable from deductions: *consequentia mirabilis* introduces new grounds for asserting formulas, and so they need to be balanced by the elimination rules for the constants of the logic.

For the time being, the following definition of 'maximal segment' is suf-

[9] C may of course be atomic. But we need not consider this case here, as we are no trying to give a proof-theoretic semantics for atomic formulas, but only for the logical constants.

ficient: A *maximal segment* is a sequence of formulas $A_1 \ldots A_n$ such that A_1 is conclusion of an introduction rule, A_n is major premise of an elimination rule and for each A_i, $1 \leq i < n$, A_i is minor premise of the form of ∨E or ∃E, i.e. rules which require collateral deductions of minor premises C which are also the conclusion of the rule, or premise of a structural rule and A_{i+1} is its conclusion. The context in which a maximal segment occurs, *i.e.* the segment plus the premises and conclusions of the rules involved in giving rise to it, may be called a *local maximum.*

Normalisation is a process involving the removal of maximal formulas as well as of maximal segments from deductions. Rules for a constant Ξ of a logic L that fulfil the criterion that such removals are possible may be said to *normalise in* L. A deduction that does not contain any maximal formulas and maximal segments is said to be *in normal form.* That any deduction of a logic L can be transformed into one in normal form is captured by the *normalisation theorem* for L. If such a theorem can be proved for a logic, it is *proof-theoretically justified*, or *justified* for short. The constants of such a logic may also be called *proof-theoretically justified.*

Consider the operator $\Box$ governed by the following introduction and elimination rules:

$$\Box I: \ \frac{X \ \vdash \ A}{X \ \vdash \ \Box A} \qquad \Box E: \ \frac{X \ \vdash \ \Box A}{X \ \vdash \ A}$$

where in $\Box I$ every formula on X is of the form $\Box B$.

There is a reduction procedure for local peaks with $\Box$:

$$\begin{array}{c} \Pi \\ \dfrac{\dfrac{X \vdash A}{X \vdash \Box A}}{X \vdash A} \\ \Sigma \end{array} \quad \text{may be reduced to} \quad \begin{array}{c} \Pi \\ X \vdash A \\ \Sigma \end{array}$$

This and the following examples ignores the possibility that $\Box$I may not be followed directly by $\Box$E, but instead there may be a number of applications of structural rules in between. Taking this complication into account is unnecessary for the purposes of my account. It will be treated properly in the formal part.

There also is a reduction procedure for local maxima with $\Box$ in the collateral deductions of $\vee$:

$$
\begin{array}{c}
\dfrac{
Y \vdash A \vee B
\qquad
\begin{array}{c}
\Pi \\
\dfrac{X \vdash C}{X \vdash \Box C} \\
\Theta \\
X''(A) \vdash \Box C
\end{array}
\qquad
\begin{array}{c}
\Sigma \\
\dfrac{X' \vdash C}{X' \vdash \Box C} \\
\Xi \\
X''(B) \vdash \Box C
\end{array}
}{
\dfrac{X''(Y) \vdash \Box C}{X''(Y) \vdash C}
} \\
\Upsilon
\end{array}
$$

may be reduced to

$$
\begin{array}{c}
\dfrac{
Y \vdash A \vee B
\qquad
\begin{array}{c}
\Pi \\
\dfrac{X \vdash C}{X \vdash \Box C} \\
\Theta \\
\dfrac{X''(A) \vdash \Box C}{X''(A) \vdash C}
\end{array}
\qquad
\begin{array}{c}
\Sigma \\
\dfrac{X' \vdash C}{X' \vdash \Box C} \\
\Xi \\
\dfrac{X''(B) \vdash \Box C}{X''(B) \vdash C}
\end{array}
}{
X''(Y) \vdash C
} \\
\Upsilon
\end{array}
$$

Should we conclude that $\Box$ is a proof-theoretically justified constant? No. Because it is meaningless to ask this question in isolation from a formal system. Notice that the status of the operator $\Box$ that one might be inclined to impose on it in the face of the existence of reduction procedures is jeopardized if one takes $\Box$ to be part of a logic that also has an implication connective, say intuitionist logic extended by $\Box$. The trouble is that in such a system local peaks with $\supset$ may not level:

$$\dfrac{\dfrac{\begin{matrix}\Pi \\ X, A \vdash B\end{matrix}}{X \vdash A \supset B} \qquad \begin{matrix}\Sigma \\ Y \vdash A\end{matrix}}{\begin{matrix}X, Y \vdash B \\ \Xi\end{matrix}}$$

may *not* be reducible to

$$\begin{matrix}\Sigma \\ Y \vdash A \\ = = = = = \\ \Pi[A/Y] \\ X, Y \vdash B \\ \Xi\end{matrix}$$

Here the second deduction is to be understood as the deduction resulting from the first by replacing all the formulas A on the branch in Π starting in the formula A of the conclusion of Π $X, A \vdash B$ by Y, and appending Σ to the initial consecutions in which the branch ends (*i.e.* which had A in their antecedents, so that after the replacement they became $Y \vdash A$). The reason why the reduction procedure is not generally applicable is that there may be formulas amongst Y not of the form $\Box B$, such that applications of $\Box I$ in $\Pi[A/Y]$ cease to be correct.

Consider a formulation of classical logic without a primitive implication connective, where negation is governed by the rules ¬I, ¬E and *consequentia mirabilis*:

$$\frac{X, A \vdash \bot}{X \vdash \neg A} \qquad \frac{X \vdash \neg A \qquad X \vdash A}{X \vdash \bot} \qquad \frac{X, \neg A \vdash \bot}{X \vdash A}$$

If $\neg E$ is applied directly after $\neg I$, this constitutes a local peak. It can be levelled by a reduction procedure very similar to the one for $\supset$. If $\Box$ is added to the logic, a similar problem occurs as in the case of intuitionist logic: local peaks with $\neg$ may no longer be removable, as applications of $\Box I$ above the maximal formula may cease to be correct.

On the other hand, as a little exercise, notice that in the following deduction $\Box A$ counts as a maximal formula, which can be removed:

$$\begin{array}{c} \Pi \\ \dfrac{\dfrac{X, \neg \Box A \vdash \bot}{X \vdash \Box A}}{X \vdash A} \\ \Sigma \end{array}$$

The maximal formal can be removed by replacing the construction by the following:

$$\begin{array}{c} \dfrac{\dfrac{\neg A \vdash \neg A \qquad \dfrac{\Box A \vdash \Box A}{\Box A \vdash A}}{\neg A, \Box A \vdash \bot}}{\neg A \vdash \neg \Box A} \\ = = = = = = = = = \\ \Pi[\neg \Box A / \neg A] \\ \dfrac{X, \neg A \vdash \bot}{X \vdash A} \\ \Sigma \end{array}$$

Any application of $\Box I$ in Π remains correct, as there can't be one in branches ending in $\neg \Box A \vdash \neg \Box A$, which are the only ones affected by the replacement.

It is possible to formulate **S4** in a different way, so that normalisation is possible. Instead of restricting the list on the left of the turnstile in $\Box I$, we can incorporate them into the form of the rule:[10]

$$\frac{X_1 \vdash \Box A_1 \qquad \ldots \qquad X_n \vdash \Box A_n \qquad \Box A_1 \ldots \Box A_n \vdash B}{X_1 \ldots X_n \vdash \Box B}$$

The elimination rule stays the same. A local peak with $\Box$ looks like this:

$$\frac{\begin{array}{c}\Sigma_1\\ X_1 \vdash \Box A_1\end{array} \qquad \ldots \qquad \begin{array}{c}\Sigma_n\\ X_n \vdash \Box A_n\end{array} \qquad \begin{array}{c}\Pi\\ \Box A_1 \ldots \Box A_n \vdash B\end{array}}{\begin{array}{c}X_1 \ldots X_n \vdash \Box B\\ \Xi\end{array}}$$

It can be levelled by replacing it with the following construction:

$$\begin{array}{c}\Sigma_1 \qquad \ldots \qquad \Sigma_n\\ =\,=\,=\,=\,=\,=\,=\,=\,=\,=\,=\,=\,=\,=\,=\\ \Pi[\Box A_1/X_1 \ldots \Box A_n/X_n]\\ X_1 \ldots X_n \vdash \Box B\\ \Xi\end{array}$$

To explain once more the notation, this is to be read as saying that we replace the formulas $\Box A_i$ on the branches starting with those formulas in the conclusion $\Box A_1 \ldots \Box A_n \vdash B$ of Π by X_i and append Σ_i to the initial consecutions in which the branches end. With these rules, deductions in classical as well as in intuitionist **S4** normalise.

Standard rules for **S4** possibility are the following:

$$\frac{X \vdash B}{X \vdash \Diamond B} \qquad \frac{X \vdash \Diamond B \qquad Y, B \vdash \Diamond C}{X, Y \vdash \Diamond C}$$

[10] This transposes rules given by Biermann and De Paiva (2000) to the present formalism. See also Prawitz (1965) for various formulations of modal logics that normalise.

Here all formulas on Y are of the form $\Box B$. These rules pose similar problems for normalisation as the rules for **S4** necessity. But the elimination rule can also be reformulated so as to incorporate the restrictions on the list in the form of the rule:

$$\frac{X \vdash \Diamond B \qquad Y_1 \vdash \Box A_1 \quad \dots \quad Y_n \vdash \Box A_n \qquad \Box A_1 \dots \Box A_n, B \vdash \Diamond C}{X, Y_1 \dots Y_n \vdash \Diamond C}$$

Deductions can then be normalised, the verification of which I leave as an exercise.

We can give similar rules for **S5** by relaxing the restrictions on the shapes of formulas. Instead of using formulas $\Box A_i$ and $\Diamond C$ in the premises of $\Box I$ and $\Diamond E$, it suffices to require that they are *modally closed*, i.e. every propositional variable is in the scope of a modal operator (hence $\bot$ and $\top$ are modally closed).

These new rules for $\Box I$ and $\Diamond E$ are not harmonious according to the definition I'll give in section 2.3. They are neither rules of type one nor of type two, as they are specified in section 2.2. But they are nearly enough. Thus it is worth allowing that rules count as harmonious also if, although they are not of type one or type two, nonetheless derive from such rules in such a way that they incorporate restrictions on lists in rules of type one or type two into their shape by adding further premises.

We can then can conclude that **S4** and **S5** count as a proof-theoretically justified logics, at least from a purely formal perspective. All the rules would be in harmony and deductions normalise. Philosophically, there remains the question whether a rule can really be said to specify the meaning of a connective if this connective occurs in the premises of the rule, but at least there seems to be no formal obstacle to accepting **S4** and **S5** that would arise if some detours were not eliminable.

1.3.4 Stability

According to Dummett, the rules for intuitionist logic are all stable. Contrast this with Dummett's example of unstable rules: the rules for disjunction in quantum logic $\veebar$. $\veebar$ has the same introduction rules as intuitionist disjunction, but the elimination rule differs in that a restriction is imposed on its application, such that the collateral deductions of the minor premises C must not depend on any hypotheses other than A and B respectively:

$$\frac{X \vdash A \veebar B \qquad A \vdash C \qquad B \vdash C}{X \vdash C}$$

In the context of quantum logic, local peaks with $\veebar$ may be levelled and thus the rules are in harmony, according to Dummett. However, if $\veebar$ is added to intutionist logic, then quantum disjunction collapses into intuitionist disjunction and the unrestricted elimination rule is derivable for $\veebar$:

$$\frac{\dfrac{X \vdash A \veebar B \qquad \dfrac{A \vdash A}{A \vdash A \vee B} \qquad \dfrac{B \vdash B}{B \vdash A \vee B}}{X \vdash A \vee B} \qquad Y(A) \vdash C \qquad Y(B) \vdash C}{Y(X) \vdash C}$$

Dummett explains that this is due to a lack of stability between introduction and elimination rules of quantum disjunction (*LBM* 290). Of course the question may arise, why do we blame quantum disjunction rather than intuitionist disjunction? But my aim here is not to challenge Dummett's characterisation of the situation, but only to extract from it some suitable formal criteria for stability from what he has to say about it.[11]

[11]Notice again how this example indicates Dummett's tendency to look for a notion of stability applicable to the form of rules of inference, not relative to particular formal

The crucial feature of the construction that shows that quantum disjunction collapses into intuitionist disjunction in the presence of the latter's elimination rule is that a *maximal segment* occurs in it which cannot be removed. In intuitionist logic, maximal segments can be removed. But suppose the reduction procedure for maximal segments is applied to the above construction:

$$\dfrac{X \vdash A \veebar B \qquad \dfrac{\dfrac{A \vdash A}{A \vdash A \vee B} \quad Y(A) \vdash C \quad Y(B) \vdash C}{Y(A) \vdash C} \qquad \dfrac{\dfrac{B \vdash B}{B \vdash A \vee B} \quad Y(A) \vdash C \quad Y(B) \vdash C}{Y(B) \vdash C}}{Y(X) \vdash C}$$

The problem here is that the last step need not be a correct application of $\veebar E$, as the restrictions on the collateral deductions of C from A and from B may not be fulfilled. Hence in a logic containing both intuitionist and quantum disjunction some maximal segments may not be removable from deductions.

This provides material for a new approach to stability. Unfavourable restrictions on applications of rules can jeopardize normalisation if the restrictions prevent local maxima or local peaks from being removable. Thus I suggest the following definition of stability: the rules governing a connective are *stable* if they are harmonious and contain no restrictions on the application of the rules. As there are no restrictions on the lists of formulas on which premises depend, stable rules guarantee a maximal amount of context independence: the context in which such rules are applied in deductions is irrelevant when it comes to removing local peaks and local maxima; it's just a matter of rearranging the deduction.

systems: what else could be the point of considering the addition of quantum disjunction to intuitionist logic?

2 The Formal Theory

2.1 Definitions

2.1.1 Languages

My aim in the following is to discuss different logics with different languages and different deductive systems in a very general way. I have found Polish Notation the most suitable for this purpose—it does not sacrifice perspicuity for the use of brackets. But the reader may be relieved to hear that I'm going to take the liberty to revert to standard notation when discussing examples. Languages consists of

i) countably many *individual variables*: $x_1, x_2 \ldots y_1, y_2 \ldots z_1, z_2 \ldots$
ii) countably many *individual parameters*: $a_1, a_2 \ldots b_1, b_2 \ldots c_1, c_2 \ldots$
iii) countably many *function symbols*, each being n-ary for some $n \in \omega$: $f_1^n, f_2^n \ldots g_1^n, g_2^n \ldots h_1^n, h_2^n \ldots$
iv) countably many *predicate symbols*, each being n-ary for some $n \in \omega$: $F_1^n, F_2^n \ldots G_1^n, G_2^n \ldots H_1^n, H_2^n \ldots$
v) a fixed number of *quantifiers*, each being n-ary-o-ary for some $n, o \in \omega$: $\Pi^{n,o}, \Sigma^{n,o}, \Xi^{n,o} \ldots$

In appropriate circumstances, super- and subscripts may be dropped for convenience. Predicates F_i^n, quantifiers $\Xi^{n,o}$, functions f_j^n where $n = 0$ cover propositional constants, connectives and individual constants, respectively. A quantifier $\Pi^{0,0}$ is a *nullary connective* or *propositional constant*, e.g. *falsum* $\bot$ and *verum* $\top$. A quantifier $\Pi^{0,1}$ is a *unary connective*, e.g. negation $\sim$, necessity $\Box$, possibility $\Diamond$. A quantifier $\Pi^{0,2}$ is a *binary connective.* Typical binary connectives are implications $\supset$ and $\rightarrow$, conjunction $\wedge$, disjunction $\vee$ and the Sheffer stroke $|$. Here are some examples for $n \neq 0$, to which the term 'quantifier' applies more naturally. A quantifier $\Xi^{1,1}$ is a *unary-unary quantifier*, e.g. the universal quantifier $\forall$, the existential quantifier $\exists$, 'not all' and 'there is no'. A quantifier $\Xi^{1,2}$ is a *unary-binary quantifier*, e.g. Russell's formal implication $\supset_x$. $\Xi^{2,1}$ is a *binary-unary quantifier*, e.g. 'For all x there is a y'. $\Xi^{2,2}$ is a *binary-binary quantifier*, e.g. binary formal implication $\supset_{x,y}$.

The two species of formal implications given as examples above are not normally treated as primitive connectives. Rather, they would be defined as $\forall x(Fx \supset Gx)$ and $\forall x \forall y(Rxy \supset Sxy)$ respectively. In a natural deduction framework, however, it is not difficult to give introduction and elimination rules for these formal implications. This would appear to provide a rationale for treating them as concepts the meanings of which are given by rules of inference rather than definitions. The question now arises: should connectives that may be defined by rules of inference be treated as primitive, such that it is, so to speak, a logical discovery that they are equivalent to certain complex expressions, or should we opt for rather more austere foundations and seek to identify a smaller number of connectives as the primitive ones in terms of which everything else is defined? The latter option is connected to a formal problem, namely whether a set of connectives of a logic has the property that every connective you might wish to add to it is definable in terms of connectives in the set. This issue is taken up in section 2.5.9.

Sometimes certain predicate symbols – in particular this one: '=' – get special treatment and are classified as logical constants. I shall not discuss identity and other such cases, an omission justified by the tradition of Gentzen and Prawitz. A case has been made by Stephen Read that identity can be given a proof-theoretic semantics.[12]

A *pseudo-term* is the following:

1. An individual parameter is a pseudo-term.
2. An individual variable is a pseudo-term.
3. A function symbol f^n_m followed by n pseudo-terms is a pseudo-term.
4. Nothing else is a pseudo-term.

A *term* is a pseudo-term with no variables.

[12]'Identity and Harmony', Analysis 64 (2004), 113-119.

A *pseudo-formula* is the following:

1. A predicate symbol F^n_m followed by n pseudo-terms is a pseudo-formula.
2. Where $\Xi^{n,o}$ is a quantifier, $\Xi^{n,o}$ followed by n individual variables followed by o pseudo-formulas is a pseudo-formula.
3. Nothing else is a pseudo-formula.

If $\Xi^{n,m}x_1 \ldots x_n A_1 \ldots A_m$ is a pseudo-formula, then $A_1 \ldots A_m$ are in the scope of $\Xi^{n,m}$. A variable x_i is said to occur *bound in* a pseudo-formula A *by* a quantifier $\Xi^{n,o}$ occurring in it iff either x_i is one of the variables directly following $\Xi^{n,o}$, or x_i occurs in a pseudo-formula in the scope of $\Xi^{n,o}$ which is not in the scope of any other quantifier immediately followed by x_i. A variable occurs *bound* in A if it is bound by some quantifier. Otherwise, the variable is *free*.

A *well-formed formula* is a pseudo-formula with no free variables. An *atomic formula* is a predicate symbol F^n_m followed by n terms or a quantifier $\Pi^{0,0}$. The definition of *subformula* of a formula A is routine, but notice that a formula containing quantifiers, like $\forall x Fx$ has infinitely many subformulas. For instance, amongst the subformulas of $\forall x Fx$ there are, *e.g.*, Fa and Ft.

$A[t/u]$ denotes the result of replacing all occurrences of the pseudo-term t in A by the pseudo-term u. If u is a variable x, it is assumed that t is not in the scope of a quantifier binding x. If t is a variable x, then u is replaced for all free occurrences of x. $A[\overline{t}/\overline{u}]$ denotes the result of replacing the pseudo-terms $\overline{t} = t_1 \ldots t_i$ with the pseudo-terms $\overline{u} = u_1 \ldots u_i$ simultaneously in A, where it is understood that both sequences have the same number of elements.

Term-forming symbols are expressions that form expressions that denote objects from other expressions, e.g. the abstraction operator λ, Hilbert's ϵ and the description operator $\imath$. They could also be added to the languages. There are important philosophical issues connected to such expressions, e.g. whether it makes sense at all to name a function or a property. A proof-theoretic semantics for such expressions should provide an interesting approach to these questions, but I won't consider this possibility here.[13]

Groupings, Lists and Sublists What I call *lists* are structured collections of well-formed formulas. Lists are formed by *grouping* formulas in specific ways. I'm calling the operations used to this purpose *groupings.* As I restrict consideration to binary ones, I'll denote groupings by commas or semi-colons, possibly with subscripts: $, \, ,_1 \, ,_2 \, , \ldots ;_1 \, ;_2 \ldots$. The list on which there are no formulas – *the empty list* – is normally marked by an empty space, but it is convenient to disallow empty spaces and instead to employ a symbol designating the empty list: 0. Thus a logic L is required to have the symbol for the empty list 0 and fixed, finite number of groupings $;_1 \ldots ;_n$, which are used together with variables $\xi_1, \xi_2 \ldots$ to define the notion of a *list-schema of* L:

1. 0 is a list-schema of L.
2. The wff of L and the variables are list-schemata of L.
3. If X and Y are list-schemata and $;_i$ is a grouping of L, then $(X;_i Y)$ is a list-schema of L.
4. Nothing else is a list-schema of L.

A *list* of L is a list-schema without variables. The notion of a *sub-list-schema* of a list X may be defined thus:

1. X is a sub-list-schema of X.
2. If $Y;_i Z$ is a sub-list-schema of X, then the sub-list-schema of Y and Z are sub-list-schema of X.
3. Nothing else is a sub-list-schema of X.

For the special case of lists I employ the less cumbersome term *sub-list.*

If $Y(\xi_h \ldots \xi_j)$ and $X_h \ldots \mathrm{X}_j$ are list-schemata, then $Y(\xi_h/X_h \ldots \xi_j/X_j)$ is the result of replacing each variable $\xi_h \ldots \xi_j$ by the lists $X_h \ldots X_j$. If

[13] For a discussion see Neil Tennant: 'A General Theory of Abstraction Operators', *Philosophical Quarterly* 54 (2004), 105-133, and Peter Milne: 'Existence, Freedom, Identity, and the Logic of Abstractionist Realism', *Mind* (2007), 23-53.

$X_h \dots X_j$ are lists, then the resulting list $Y(\xi_h/X_h \dots \xi_j/X_j)$ is an *instance* of the list-schema $Y(\xi_h \dots \xi_j)$. This allows the alternative definition 'Y_1 is a sub-list-schema of X iff there is some list-schema $Y_2(\xi_i)$ such that $X = Y_2(\xi_i/Y_1)$'.

If X is a sub-list-schema of Y we may say that X *occurs on* Y. Hence the empty list is not a sub-list of every list, but only of those on which 0 occurs.

The notation $Z(Y)$ and $V(X;Y)$ indicates that Y and $X;Y$ are sub-list-schemata of Z and V, resp.. Accordingly, V and Z may not add anything to Y and $X;Y$, *i.e.* $Z(Y)$ may just be Y and $V(X;Y)$ may just be $X;Y$. Y and $X;Y$ however must always be there, as what fails to exist cannot be a sublist of anything.

In the special case where no formulas occur on list-schemata, *i.e.* they consists only of groupings and variables, I shall denote them by upper case Greek letters, *e.g.* Θ, possibly followed by their variables as in $\Theta(\xi_h \dots \xi_j)$, where I assume that $\xi_h \dots \xi_j$ are *all and only* the variables on Θ. If a list $X = \Theta(\xi_h/X_h \dots \xi_j/X_j)$, Θ may be called the *salient substructure* of X. In such a case I shall write more briefly $\Theta(X_h \dots X_j)$. The notation will be used if two lists X and Y are given such that for sublists $X_1 \dots X_n$ of X and $Y_1 \dots Y_n$ of Y, there is a listschema $\Theta(\xi_1 \dots \xi_n)$ such that $X = \Theta(\xi_1/X_1 \dots \xi_n/X_n)$ and $Y = \Theta(\xi_1/Y_1 \dots \xi_n/Y_n)$. In such a case X and Y have a *common substructure.*

2.1.2 Deductive Systems

Consecutions Following Anderson & Belnap[14], I call a pattern of the form

$$X \vdash B$$

[14] *Entailment. The Logic of Relevance and Necessity*, Vol. 1 (Princeton 1974), §7.2.

a *consecution*, as it is convenient to have the word 'sequent', which is usually employed as the translation of Gentzen's *Sequenz*, at one's disposal for other purposes. What is to the left of the turnstile $\vdash$ is the *antecedent* of the consecution and what is to its right is the *succedent* or *consequent.* We may say that X and B are *in* the antecedent and succedent, resp., of the consecution $X \vdash B$.

Gentzen provides the following interpretation for consecutions $X \vdash B$, where he uses an arrow where I use the turnstile: 'to each (formalised) assertion $\mathfrak{B}$ the (formalised) assumptions $\mathfrak{A}_1 \ldots \mathfrak{A}_\mu$ on which it depends are added in the following form:

$$\mathfrak{A}_1 \ldots \mathfrak{A}_\mu \to \mathfrak{B};$$

to be read: $\mathfrak{B}$ holds under the assumptions $\mathfrak{A}_1 \ldots \mathfrak{A}_\mu$.'[15]

Structural Rules are divided into two categories. First, there are structural rules for the groupings. Secondly, there are structural rules for the empty list. Using the Θ-notation for structures of lists, structural rules allow replacing a sublist X of $Y(X)$ with substructure $\Theta(X_1 \ldots X_l)$ by another list Z with substructure $\Psi(Z_1 \ldots Z_m)$, *i.e.* they have the general form:

$$\frac{Y(\Theta(X_1 \ldots X_l)) \ \vdash \ A}{Y(\Psi(Z_1 \ldots Z_m)) \ \vdash \ A}\,\mathrm{S}$$

When using this notation it is understood that $\Theta(X_1 \ldots X_l)$ and $\Psi(Z_1 \ldots Z_m)$ are replaced for the same variable ξ in $Y(\xi)$. Although this may be a somewhat unnatural use of terminology, it will be convenient for later purposes to call the upper A the *main premise* of the structural rule and the lower one its *conclusion*; we say, more naturally, that a structural rule is *applied*

[15] Gerhard Gentzen: 'Die Widerspruchsfreiheit der reinen Zahlentheorie', *Mathematische Annalen* 112 (1936), 493-565, p.512.

to a list. To save space structural rules may be written more concisely in the following way: $\Theta(X_1 \ldots X_l) \Leftarrow \Psi(Z_1 \ldots Z_m)$, meaning that if $\Theta(X_1 \ldots X_l)$ occurs on a list, it may be replaced by $\Psi(Z_1 \ldots Z_m)$.

The general form of a structural rule given in the last paragraph is too liberal for the purposes of modelling acceptable reasoning. It allows there to be a structural rule that licenses the replacement of any list by any other list. Undoubtedly this is not a very good rule to have in a logic. Structural rules for formal systems are motivated by the kinds of collections of assumptions the logic is envisaged to employ. What is needed are some restrictions. Call a structural rule $X \Leftarrow X'$ *reasonable* if satisfies the following requirements: (i) No restrictions are made on $Y(\xi)$, *i.e.* it is an arbitrary list-schema (i.e. we only consider structural rules where the $\Leftarrow$-notation is applicable); (ii) No restrictions are imposed on the shapes of formulas occurring on the lists X and X', *i.e.* the structural rules could be formulated by using list-schemata with only variables on them; (iii) it maps every formula in the antecedent of the premise of the rule onto a *unique* formula of the same shape in the antecedent of its conclusion.

In the following I shall only consider reasonable structural rules. They are thus restricted to those that allow the *reordering* of lists of formulas on lists, the *deletion* of duplicate lists of formulas and the *addition* of lists of formulas to lists. The so-called Mingle rule $X \Leftarrow X; X$ is not reasonable, as it fails to map formulas in the antecedent of the premise onto unique formulas in the antecedent of the conclusion. Here are some examples of structural rules commonly found in the literature:[16]

[16] *Cf.* for instance Greg Restall: *Substructural Logics* (London, New York: Routledge 2000).

Associativity	$X;(Y;Z) \Leftarrow (X;Y);Z$
Twisted Associativity	$X;(Y;Z) \Leftarrow (Y;X);Z$
Converse Associativity	$(X;Y);Z \Leftarrow X;(Y;Z)$
Strong Commutativity	$(X;Y);Z \Leftarrow (X;Z);Y$
Weak Commutativity	$X;Y \Leftarrow Y;X$
Strong Contraction	$(X;Y);Y \Leftarrow X;Y$
Weak Contraction	$X;X \Leftarrow X$
Thinning	$X \Leftarrow X;Y$

The other kind of structural rules concern the empty list 0. They codify how the empty list together with a grouping may be added to or deleted from lists. These are common rules for the empty list and a grouping ;:

Left Addition	$+0$	$X \Leftarrow 0;X$
Left Subtraction	-0	$0;X \Leftarrow X$
Right Addition	$0+$	$X \Leftarrow X;0$
Right Subtraction	$0-$	$X;0 \Leftarrow X$

Just as I shall not assume logics to have particular structural rules, I shall not assume them to have specific rules for the empty list either. The only restriction is that the structural rules of each logic ensure that the empty list deserve its title. This is captured by the following definition. An empty list 0 may be called *genuine* in a logic L iff the following two conditions hold: (i) for any grouping ; of L, $0;0 \Leftarrow 0$ is an admissible structural rule of L, which is to say that a list of 0s just is an empty list; and (ii) for any list with structure $\Theta(Z\ X_1 \ldots X_m)$ as used in an introduction rule of type 1 for a connective of L (*cf.* section 2.2.2), $\Theta'(X_1 \ldots X_m) \Leftarrow \Theta(0\ X_1 \ldots X_m)$ is an admissible rule, where $\Theta'(X_1 \ldots X_m)$ is $\Theta(0\ X_1 \ldots X_m)$ with 0 deleted together with the grouping which combines it with one of the X_i. In other words, the empty list may be added to sublists. For instance, if $\Theta(0\ X_1 \ldots X_m)$ is $X_1;(0;X_2)$, then $\Theta'(X_1 \ldots X_m)$ is $X_1;X_2$, and if 0 is genuine in the logic, then its structural rules guarantee that the latter may be replaced by $X_1;(0;X_2)$ in deductions.

The point of the second requirement will become clearer once the general forms of harmonious rules of inference have been given.

Operational Rules for a quantifier $\Pi^{n,o}$ are divided into *introduction rules* $\Pi^{n,o}$I and *elimination rules* $\Pi^{n,o}$E. The former specify under which conditions a formula $\Pi^{n,o}x_1 \dots x_n C_1 \dots C_o$ with $\Pi^{n,o}$ as *main operator* may be derived and the latter which formulae may be derived from $\Pi^{n,o}x_1 \dots x_n C_1 \dots C_o$. If a rule is an introduction or elimination rule for $\Pi^{n,o}$ we may say that that $\Pi^{n,o}$ *figures* in the rules. Initially, I shall restrict consideration to rules in which exactly one constant figures exactly once and in which no constant occurs other than the one figuring in it. This excludes rules like distribution:

$$\frac{X \vdash A \wedge (B \vee C)}{X \vdash (A \wedge B) \vee C}$$

In order to be able to discuss logics such as classical logic, in section XY the restrictions is loosened so as to allow also for rules such *consequentia mirabilis*, in which *falsum* figures together with negation.

It is convenient to use the terms *premise* and *conclusion* ambiguously, as this allows a certain flexibility of expression. A first way of speaking is to call the *consecutions* above the line of a rule its premises and the one below is its conclusion. If the aim is to prove a normalisation theorem for a formal system, we are mainly interested in the succedents of consecutions, because this is where the maximal formulas to be removed occur, rather than in the whole consecutions. Thus the *formulas in the succedents of the consecutions* above the line may also be called the premises of the rule and the corresponding formula below the line its conclusion. Context should always disambiguate and prevent confusions.

The premise containing the constant figuring in it is the *major premise* of an elimination rule, all others being *minor premises*. Minor premises may also be called *collateral premises* and the deductions leading to the consecutions having minor premises in their succedents may be called *collateral* *deductions.* Many rules have formulas in the antecedents of their premises which do not occur in the antecedent of the conclusion: these are the *discharged assumptions* or *hypotheses* of the rule.

Deductions A *logic* consists of a language $\mathcal{L}$ and a *deductive system* or *finite set of rules* $\mathfrak{R}$, consisting of operational and structural rules of the logic plus the rule that you can always write down *initial consecutions* of the form $A \vdash A$ to get deductions started. I shall use the term *deduction rules* to cover both, structural and operational rules. In constructing deductions, deduction rules are *applied* and we may speak of *applications of rules.* In accordance with the deliberately ambiguous terminology introduced in the last section, we may say both, that operational rules are applied to *formulas* in the succedents of consecutions or to the *consecutions* which have the formulas in their succedent, and similarly in the case of structural rules, we may say both, that they are applied to consecutions or to lists in the antecedent of consecutions.

Deductions in a logic are constructed from initial consecutions by applications of deduction rules. More precisely, let L be a logic with language $\mathcal{L}$ and deductive system $\mathfrak{R}$. Then 'Π is a *deduction in* or *of* L *of the consecution* $Y \vdash B$' may be defined inductively:

1. If $A \in \mathcal{L}$, then $A \vdash A$ is a deduction (of $A \vdash A$) in L.
2. Let $\Pi_1 \ldots \Pi_n$ be deductions in L of $X_1 \vdash A_1 \ldots X_n \vdash A_n$, resp. and let

$$\frac{X_1 \vdash A_1 \ldots X_n \vdash A_n}{Y \vdash B}$$

be an application of the rule $\mathfrak{r} \in \mathfrak{R}$. Then

$$\frac{\Pi_1 \ldots \Pi_n}{Y \vdash B}$$

is a deduction (of $Y \vdash B$) in L.
3. Nothing else is a deduction of L.

Given the form of operational rules, deductions are in *stammbaumförmiger Anordnung*, as Gentzen writes—they are 'family tree shaped'.

I'm calling the formulas in the succedent of initial consecutions the *hypotheses of the deduction*, and say that hypotheses are *introduced into* deductions *by* initial consecutions. Each node in a deduction determines in the obvious way a *subdeduction* of the consecution at this node. Following Gentzen's interpretation of consecutions, the formulas in the antecedent of a consecution are the *assumptions* on which the formula in its succedent *depends*. The formula in the succedent of the bottom-most node of the deduction is the *conclusion of the deduction*, and the formulas in its antecedent I call the *assumptions of the deduction*.

Where A is the conclusion of a deduction Π and X are the assumptions of the deduction, let $\Pi(X \vdash A)$ mean 'Π *is a deduction of* A *depending on the assumptions* X' or 'Π *is a deduction of* A *from the assumptions* X'. This is of course to be understood as relative to a logic L—which logic is normally clear from the context. If it needs to be made explicit, we can adopt the notation $\Pi(X \vdash_{\mathrm{L}} A)$.

Strings and Branches A *string* ς is a sequence $A_1 \ldots A_n$ of formulas in a deduction (occurring in the succedents of consecutions) such that A_1 is either a hypothesis or a conclusion of an application of an operational rule, and every A_i, $i < n$, is premise of a structural rule, and A_n is premise of an operational rule or the conclusion of the deduction. For brevity's sake I shall speak of strings being premises and conclusions of operational rules, or being introduced as hypotheses, if this is true of their their first or last formula. We may say that $A_1 \ldots A_n$ are *on* ς.

Operational rules of the forms to be given in section 2.2 below have the property that their applications map each undischarged assumption in the antecedent of a premise of the rule onto a *unique* formula in the antecedent of the conclusion. If all the structural rules of a logic are reasonable, their applications, too, map formulas in the antecedent of a consecution in a deduction uniquely onto others in the lines below them. Call the function this observation gives rise to f_β. As the formulas and the ones they are are mapped onto by f_β are all of the same shape, we can assign each an index to distinguish them: starting with some formula A_1 in the deduction, the formulas that are mapped onto it may be denoted by A_{11}, $A_{12} \ldots A_{1n}$; the formulas mapped onto these may be denoted by A_{111}, $A_{112} \ldots A_{121}$, $A_{122} \ldots A_{1n1}$, $A_{1n2} \ldots A_{1nm}$; and so on until finally formulas in the antecedents of initial consecutions or introduced by Thinning are reached. The set of all these formulas I shall call the *branch* β *in the deduction* Π *beginning with* A_1. We may say that the formula A is *on* the branch β. Branches in deductions may be visualised in the obvious way suggested by the name. They are ordered sets with a unique bottom node, which we may call the first formula on the branch, and they end in top nodes which are formulas in the antecedent of initial consecutions or have been added by Thinning, which we may call the last formulas on the branch. More formally, where $f^n_\beta(x)$ is f_β applied n times to x, the branch beginning with A_1 is the set $\{x \mid \exists n(f^n_\beta(x) = A_1)\}$.

Consequence Relations The deduction rules of a logic L determine a *consequence relation* $\vdash_{\mathrm{L}}$. If consideration is restricted to consequence relations holding between lists and formulas of the language of L, then a very straightforward definition suffices: $X \vdash_{\mathrm{L}} A$ iff there is a deduction Π in L such that $\Pi(X \vdash A)$. For many systems, in particular those not containing any grouping for which Thinning is a structural rule, *e.g.* the relevance logic **R**, this is enough. However, for systems such as classical logic **C** and intuitionist logic **I**, it is customary to include the case where conclusions are drawn from infinitely many formulas. Only a finite number of formulas can occur on a list, so in order for it to be possible to draw conclusions from infinite collections of formulas, lists need to represent suitable finite sub-collections of these infinite collections, such as equivalence classes of lists under the relation of interderivability by structural rules of the logic in question. A suitable inclusion relation $\preccurlyeq$ needs to be defined for the collections of formulas, and then we may define $\Gamma \vdash_{\mathrm{L}} A$ iff there is a collection Δ such that $\Delta \preccurlyeq \Gamma$ and a list X which represents Δ and there is a deduction Π in L such that $\Pi(X \vdash A)$.

The theorems of a logic are those formulas which may be derived from the empty list of assumptions, *i.e.* A is a *theorem of* L iff for some deduction Π of L $\Pi(0 \vdash A)$. In this case we say that Π is a *proof* of A in L.

2.2 The General Forms of Operational Rules

2.2.1 Introduction

In this section I shall specify the general forms of *two* kinds of operational rules and the function Gentzen speaks of which provides a mapping between introduction and elimination rules. In the first case, initially it is one introduction rule that is assumed to be given, and there is a general method for reading off elimination rules from it. In the second case, conversely, initially it is one elimination rule which is assumed to be given, and there is a general method for reading off introduction rules from it. The process can be reversed, so that, for a rule of type one, we could also be given the elimination rules and read off its introduction rule from them, and for a rule of type two, we could also be given the introduction rules and read off its elimination rule from them. But in either case it needs to be specified which of the two types the rules belong to. This constitutes a departure from the received approaches: normally it is assumed that uniformly, either the introduction or the elimination rules are given, and the corresponding elimination and introduction rules are determined relative to them. To my knowledge it has not been claimed before that rules of inference come in two different forms, even though this is supported by the intuition that there is something very different about, for instance, the rules for conjunction and those for disjunction.

2.2.2 The First Type of Rules

For the first type of operational rules, we begin with an introduction rule for a constant Ξ is given and specify a general method for determining its elimination rules. An introduction rule of type one has the following form:

$$\frac{\Phi(X\ A_1 \dots A_h) \vdash B_1 \qquad \dots \qquad \Psi(X\ A_k \dots A_l) \vdash B_p}{X \vdash \Xi\,\overline{x}\,A_1 \dots A_l B_1 \dots B_p[\overline{a}/\overline{x}]}$$

where there are no formulas on X in which the parameters on the sequence $\overline{a} = a_1 \dots a_v$ occur.

Above the line, $A_1 \dots A_l$ are the assumptions discharged by an application of the rule and $B_1 \dots B_p$ are its premises. It is understood that all and only the premises and discharged assumption reoccur as immediate subformulas of the conclusion. Hence Ξ is a v-ary-$(l+p)$-ary quantifier. In constructing the conclusion of the rule from the discharged assumptions $A_1 \dots A_l$ and the premises $B_1 \dots B_p$ the convention is adopted that first comes Ξ, then the variables it binds, then the discharged assumptions in the order of their occurrence above the line from left to right, and finally the premises in the same order, where in each formula the parameters $\overline{a}$ are replaced by the variables $\overline{x}$. For each premise B_o there is a list-schema $\Theta(\zeta_1 \dots \zeta_m)$ so that the list on which B_o depends in the sub-deduction leading to it has the salient sublists $X, A_i \dots A_j$ and is $\Theta(\zeta_1/X\ \zeta_2/A_i \dots \zeta_m/A_j)$.[17] The substructure of

[17]Strictly speaking, if wff are in the object language and rules of inference of formal systems are given in the metalanguage, then the general forms of rules should be given in a meta-metalanguage, and accordingly $X, A_j \dots A_k$, B_o as used in this sentence should

the lists on which the premises depend can be different in each sub-deduction leading to a premise, hence the different uppercase Greek letters. Note that it is irrelevant which groupings $;_o \ldots ;_t$ occur in Θ and what its structure is. In giving the general forms of harmonious operational rules, we may completely abstract from these particulars.

The method for specifying the elimination rules for a constant Ξ with an introduction rule of type one is the following. To each premise of the introduction rule for Ξ, there is a corresponding elimination rule. The number of minor premises of each elimination rule is determined by the number of assumptions discharged by an application of ΞI that the premise it corresponds to depends on. The list on which the conclusion of an elimination rule depends is the list on which this premise of ΞI depends with the discharged assumptions replaced by the lists on which its minor premises depend. Thus there are p elimination rules ΞE_p, one for each premise B_o of the introduction rule, each being of the following form, where it is understood that B_o is amongst the $B_1 \ldots B_p$ and $A_i \ldots A_j$ are amongst the $A_1 \ldots A_l$ and $\bar{t}$ is a sequence of terms:

$$\frac{Z \vdash \Xi\,\overline{x}\,A_1 \ldots A_l B_1 \ldots B_p \qquad Y_i \vdash A_i[\overline{x}/\overline{t}] \qquad \ldots \qquad Y_j \vdash A_j[\overline{x}/\overline{t}]}{\Theta(Z\ Y_i \ldots Y_j) \vdash B_o[\overline{x}/\overline{t}]}$$

It is understood that the order in which $Y_i \vdash A_i[\overline{x}/\overline{t}] \ldots Y_j \vdash A_j[\overline{x}/\overline{t}]$ occur equals the order in which $A_i \ldots A_j$ occur in $\Xi\,\overline{x}\,A_1 \ldots A_l B_1 \ldots B_p$, which of course is no loss of generality as any deduction may be ordered in such a way as to fulfil this requirement.

The minor premises are the same as the assumptions discharged by an application of ΞI on which B_o depends in ΞI, only with the sequence of

in a meta-meta-metalanguage, as they are variables ranging over meta-metalanguage expressions. I reckon that making these distinctions precise in the text would not add to perspicuity, so suffice it to make this point once in a footnote.

variables $\overline{x} = x_1 \dots x_v$ simultaneously replaced by the sequence of terms $\bar{t} = t_1 \dots t_v$. $B_o[\overline{x}/\overline{t}]$ too, is just like B_o, but with variables $x_1 \dots x_v$ replaced by terms $t_1 \dots t_v$. The list on which the conclusion of ΞE_o depends is obtained from the list-schema $\Theta(\zeta_1 \dots \zeta_m)$ of the list on which B_o depends in ΞI by replacing $\zeta_1 \dots \zeta_m$ by $Z, Y_i \dots Y_j$, *i.e.* the list is $\Theta(\zeta_1/Z\ \zeta_2/Y_i \dots \zeta_m/Y_m)$.

The point of the second condition imposed on genuine empty lists in section 2.1.2 should now become clear. Its rationale is to guarantee that if each premise B_o of the introduction rule has been derived depending on $\Theta'(A_i \dots A_j)$, one can apply a structural rule to derive B_o from $\Theta(0\ A_i \dots A_j)$ and then prove a theorem with Ξ as main connective.

The process of determining the elimination rules from the introduction rules can be inverted, so that the elimination rules are given first and the introduction rule determined from it. To put it briefly, each elimination rule for a connective governed by a rule of type one determines a consecution which is a premise of its introduction rule, where the formula that is in the consequent of the consecution reappears as a premise and the formulas which are minor premises reappear as discharged assumptions.

Examples of Rules of Type One

i. *Verum*

The constant $\top$ is governed by an introduction rule with no premises:

$$X \vdash \top$$

Hence $\top$ has no elimination rule.

ii. Conjunction

The constant $\wedge$ is governed by an introduction rule with two premises and no discharged hypotheses:

$$\frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}$$

Hence $\wedge$ has two elimination rules without minor premises:

$$\frac{X \vdash A \wedge B}{X \vdash A} \qquad\qquad \frac{X \vdash A \wedge B}{X \vdash B}$$

Comment

These are the stable rules for conjunction. Some people prefer a different version of conjunction introduction:

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B}$$

If this rule is used, the levelling of local peaks with $\wedge$ needs Thinning as a structural rule for the comma. Hence whether a conjunction governed by these rules is justified depends on the structural rules of the system. The rules just given are thus, maybe surprisingly, the more general ones, as whether a local peak with $\wedge$ governed by them can be levelled is independent of the structural rules of the logic.

iii. Universal Quantification

The unary-unary quantifier $\forall x$ is governed by an introduction rule with one main premises and no discharged hypotheses:

$$\frac{X \vdash A}{X \vdash \forall x A[a/x]}$$

where a does not occur in any formula on X. Hence $\forall x$ has one elimination rule:

$$\frac{Y \vdash \forall x A}{Y \vdash A[x/t]}$$

iv. Implication

Implication $\rightarrow$ is governed by an introduction rule with one premise and one discharged hypothesis:

$$\frac{X; A \vdash B}{X \vdash A \rightarrow B}$$

Hence it has one elimination rule with one minor premise:

$$\frac{Y \vdash A \rightarrow B \qquad Z \vdash A}{Y; Z \vdash B}$$

The rules for relevant implication are the same as the rules for material implication; whether the conditional is relevant or not depends on further details of the logic, in particular whether Thinning is a structural rule for ;. Using the notation for substructures in the formulation of the rules, we could generalise them and say that a connective $\rightarrow$ is an *implication for the grouping* ; if ; is the salient grouping of the structure Θ and $\rightarrow$ is governed by rules of the form:

$$\frac{\Theta(X \ A) \vdash B}{X \vdash A \rightarrow B} \qquad\qquad \frac{Y \vdash A \rightarrow B \qquad Z \vdash A}{\Theta(Y \ Z) \vdash B}$$

Notice that there are *two* possible implications for a grouping ;, which, in the absence of the structural rule $X; Y \Leftarrow Y; X$ need not amount to the same thing: in one case $\Theta(X \quad A) = X; A$, in the other $\Theta(X \quad A) = A; X$. The first may be called the *right implication* for ;, the second its *left implication*. Substructures might thus lend themselves for the formulation a general theory of connectives, which is left for future investigation.

v. Formal Implication

The unary-binary quantifier $\supset_x$ is governed by an introduction rule with one main premise and one discharged hypothesis:

$$\frac{X, Aa \vdash Ba}{X \vdash Ax \supset_x Bx}$$

where a does not occur in any formula on X. Hence it has one elimination rule with one minor premise:

$$\frac{Y \vdash Ax \supset_x Bx \qquad Z \vdash At}{Y, Z \vdash Bt}$$

vi. Biconditional

The biconditional is governed by an introduction rule with two premises and one discharged hypothesis for each premise:

$$\frac{X, A \vdash B \qquad X, B \vdash A}{X \vdash A \leftrightarrow B}$$

Hence it has two elimination rules, each with one minor premise:

$$\frac{X \vdash A \leftrightarrow B \qquad Y \vdash A}{X, Y \vdash B} \qquad\qquad \frac{X \vdash A \leftrightarrow B \qquad Y \vdash B}{X, Y \vdash A}$$

One could generalise these rules using the substructure notation to define the notion of a *biconditional for a grouping* ;.

2.2.3 The Second Type of Rules

For the second type of operational rules, we begin with an elimination rule for a constant Ξ and specify a general method for determining its introduction rules. An introduction rule of type two has the following form:

$$\frac{Z \vdash \Xi\,\overline{x}\,D_1 \ldots D_n \qquad Y(\Phi(D_1[\overline{x}/\overline{a}] \ldots D_i[\overline{x}/\overline{a}])) \vdash E \qquad \ldots \qquad Y(\Psi(D_l[\overline{x}/\overline{a}] \ldots D_n[\overline{x}/\overline{a}])) \vdash E}{Y(Z) \vdash E}$$

where none of the parameters on the sequence $\overline{a} = a_1 \ldots a_v$ occurs in E or any formula on Y.

$\Xi\,\overline{x}\,D_1 \ldots D_n$ is the major premise of the rule, the Es above the line are its minor premises and $D_1[\overline{x}/\overline{a}] \ldots D_n[\overline{x}/\overline{a}]$ to their left are the assumptions discharged by an application of the rule. It is understood that all and only the discharged assumptions occur in $\Xi\,\overline{x}\,D_1 \ldots D_n$ (with parameters replaced by variables). Hence Ξ is a v-ary-n-ary quantifier. $D_1 \ldots D_n$ occur in $\Xi\,\overline{x}\,D_1 \ldots D_n$ in the order in which they occur as discharged assumptions in the antecedents of the minor premises from left to right through the subdeductions leading to them. There is a list-schema $Y(\xi)$ such that for each minor premise E there is a list-schema $\Theta(\zeta_j \ldots \zeta_k)$ and a collection of assumptions $D_j[\overline{x}/\overline{a}] \ldots D_k[\overline{x}/\overline{a}]$ discharged by the rule so that E depends on $Y(\xi/\Theta(\zeta_j/D_j[\overline{x}/\overline{a}] \ldots \zeta_k/D_k[\overline{x}/\overline{a}]))$, and the list on which the conclusion depends is $Y(\xi/Z)$, where Z is the list on which the major premise depends. The substructure of the lists of discharged hypotheses can be different in each sub-deduction leading to a premise, hence the different uppercase Greek letters. It is irrelevant which groupings $;_o \ldots ;_t$ Θ uses and what the structure Θ is.

The method for specifying the introduction rules for a constant Ξ with an elimination rule of type two is the following. The number of introduction rules of Ξ equals the number p of minor premises of ΞE. To each consecution having a minor premise in its succedent there corresponds an introduction rule, and the number of premises of each introduction rule is determined by the number of assumptions discharged by an application of ΞE that this minor premise depends on. The list on which the conclusion of an introduction rule depends is the sublist of the list on which the minor premise depends on which the discharged assumptions are found, with these assumptions replaced by the lists on which the premises of the introduction rule depend. Thus there are p introduction rules ΞI_p, one for each minor premise E, each being of the following form, where it is of course understood that the $D_j \ldots D_k$ are amongst the $D_1 \ldots D_n$:

$$\frac{X_j \vdash D_j[\overline{x}\,/\,\overline{t}\,] \qquad \ldots \qquad X_k \vdash D_k[\overline{x}\,/\,\overline{t}\,]}{\Theta(X_j \ldots X_k) \vdash \Xi\,\overline{x}\,D_1 \ldots D_n}$$

It is understood that the order in which the premises $D_j \ldots D_k$ occur above the line equals the order in which they occur in $\Theta(X_j \ldots X_k) \vdash \Xi\,\overline{x}\,D_1 \ldots D_n$, which of course is no loss of generality as any deduction may be ordered in such a way as to fulfil this requirement.

The premises are the same as the assumptions discharged by an application of ΞE except that the sequence of free variables $\overline{x} = x_1 \ldots x_v$ are replaced by the sequence of terms $\overline{t} = t_1 \ldots t_v$. The list on which the conclusion of an introduction rule for Ξ depends is constructed from the list schema $\Theta(\zeta_j \ldots \zeta_k)$ of the list on which the assumptions $D_j[\overline{x}/\overline{a}] \ldots D_k[\overline{xa}]$ discharged by ΞE are by replacing $\zeta_j \ldots \zeta_k$ by $X_j \ldots X_k$, *i.e.* the list is $\Theta(\zeta_j/X_j \ldots \zeta_k/X_k)$.

The process can be inverted. To be brief, the number of introduction rules for a connective governed by rules of type two determines the number of collateral deductions, where each premise corresponds to a discharged assumption in a collateral deduction.

Examples of Rules of Type Two

i. *Falsum*

The constant $\bot$ is governed by an elimination rule without minor premises:

$$\frac{X \ \vdash \ \bot}{Y(X) \ \vdash \ A}$$

Hence $\bot$ has no introduction rule.

Comment.

It might be thought that as there are no collateral deductions, $Y(X)$ should be just X. Reflection shows that this is not so. An elimination rule of type two says that if a number p of deductions of a formula E from a list-schema $Y(\xi)$ are given, where the variable is replaced by a list which groups some of the hypotheses $D_1 \ldots D_n$ discharged by an application of the rule, then we may derive E from $Y(Z)$, where Z is the list on which the major premise depends. *falsum* has no subformulas, so there are no collateral deductions and discharged hypotheses. So $Y(\xi)$ is arbitrary.

ii. Existential Quantification

The unary-unary quantifier $\exists x$ is governed by an elimination rule with one minor premise and one discharged hypothesis:

$$\frac{X \vdash \exists x F x \qquad Y(A[x/a]) \vdash C}{Y(X) \vdash C}$$

where a does not occur in C or any formula on Y. Hence it has one introduction rule:

$$\frac{Z \vdash A[x/t]}{Z \vdash \exists x A}$$

iii. Truth Constant t for the Empty List

The constant **t** is governed by an elimination rule with one minor premise where 0 occurs in the place of a discharged assumption:

$$\frac{X \vdash \mathbf{t} \qquad Y(0) \vdash C}{Y(X) \vdash C}$$

Hence **t** has one introduction rule with no premises:

$$0 \vdash \mathbf{t}$$

Comment

t is a proposition made true by the logic alone, but different from $\top$. It is worth noting here that there are no rules of type one or type two that govern the corresponding proposition expressing logical falsity **f**, as used in relevant logics. Rules for **f** need to appeal to negation. Proof-theory seems to be biased towards truth and neglects falsity.

vi. Fusion

The constant $\times$ is governed by an elimination rule with one minor premise and two discharged hypotheses:

$$\frac{X \vdash A \times B \qquad Y(A; B) \vdash C}{Y(X) \vdash C}$$

Hence it has one introduction rule with two premises:

$$\frac{X \vdash A \qquad Y \vdash B}{X; Y \vdash A \times B}$$

$\times$ is closely connected to the grouping ; independently of the structural rules used in a logic, as $Y(A;B) \vdash C$ iff $Y(A \times B) \vdash C$. Left to right follows easily by $\times E$, right to left by $\times I$ and Cut, which is shown to be an admissible rule of the systems under consideration in section 2.5.8. This is useful in characterising a general notion of a conjunction. Using the Θ-notation we can say that a connective κ is a *conjunction for the grouping* ; of a logic if ; is the salient grouping in Θ and κ is governed by the rules:

$$\frac{X \vdash \kappa AB \qquad Y(\Theta(A\ B)) \vdash C}{Y(X) \vdash C} \qquad\qquad \frac{X \vdash A \qquad Y \vdash B}{\Theta(X\ Y) \vdash \kappa AB}$$

There are *two* possible conjunctions for a grouping ; which, in the absence of the structural rule $X;Y \Leftarrow Y;X$, need not amount to the same thing: in one case $\Theta(X\ Y) = X;Y$, in the other $\Theta(X\ Y) = Y;X$. The first may be called the *right conjunction* for ;, the second its *left conjunction*, mirroring the terminology for implications. For the left conjunction $\times_l$ for the semi-colon, we have $Y(B;A) \vdash C$ iff $Y(A \times_l B) \vdash C$.

Notice that $\wedge$, on the other hand, is only closely connected to a grouping in special cases, such as **I** and **C**. where Thinning is a structural rule.

v. Disjunction

The constant $\vee$ is governed by an elimination rule with two minor premises, each with one discharged assumption:

$$\frac{X \ \vdash \ A \vee B \qquad Y(A) \ \vdash \ C \qquad Y(B) \ \vdash \ C}{Y(X) \ \vdash \ C}$$

Hence $\vee$ has two introduction rules with one premise each:

$$\frac{X \ \vdash \ A}{X \ \vdash \ A \lor B} \qquad\qquad \frac{X \ \vdash \ B}{X \ \vdash \ A \lor B}$$

2.2.4 Negation

The only negations covered so far are negations defined in terms of implication and *falsum*, i.e. $\neg A =_{def.} A \to \bot$. For these negations, *ex contradictione quodlibet* $\neg A; A \vdash B$ is valid, where $\to$ is an implication for ;. This is not suitable for relevance logics. To formalise negations for relevant logics, we need to lift the restriction that only one constant can figure in a rule and introduce a symbol **f**. The introduction rule for negation is then almost a rule of type one, except that **f** occurs in it together with $\neg$:

$$\frac{\Theta(X \ A) \vdash \mathbf{f}}{X \vdash \neg A}$$

The rule has one premise and one discharged assumption, hence it has one elimination rule with two premises:

$$\frac{Z \vdash \neg A \qquad Y \vdash A}{\Theta(Z \ Y) \vdash \mathbf{f}}$$

We can say that $\neg$ is a negation for the salient grouping of Θ.

If **f** is considered to be a proposition, it's intended interpretation is as a logical falsehood. Whether it amounts to the same as $\bot$ depends on the further features of the logic, in particular whether Thinning is a structural rule for the salient grouping of Θ. Notice that $\bot$ can never be a maximal formula, as it has no introduction rules. Similarly, I consider **f** as never being a maximal formula: the rules above are not introduction and elimination rules for **f**, but only for $\neg$. In fact, then, **f** is not governed by any introduction and elimination rules at all. Thus a case could be made that **f** is not a proposition at all, but a punctuation mark that is only needed to codify the use of negation.[18] If **f** is not a proposition, it's meaning would of course not have to be given by a proof-theoretic semantics, as it doesn't have a meaning. It would be possible not to use the symbol **f** at all, but rather allow for an empty space to the right of the turnstyle in negation rules. But just as we introduced a special symbol for the empty list, it is natural to introduce a symbol for empty spaces on the right of the turnstyle, too. We could have used 0 for that purpose, but **f** is customary in the literature on relevance logic. Using 0 or an empty space instead of **f** would of course prevent us from having this special logical falsehood that **f** is supposed to be amongst the assumptions of consecutions. Notice that $0 \vdash 0$ or $0 \vdash \mathbf{f}$ are not initial consecutions, as only consecutions of the form $A \vdash A$ are initial where A is a formula. Viewing **f** as a punctuation mark or an empty space to the right of the turnstyle rather than a formula simplifies things at a later stage, so this is the course I'm going to take. Hence $\mathbf{f} \vdash \mathbf{f}$ is not an initial consecution either.

From the philosophical perspective, viewing **f** as not being a proposition at all would also get around the point that, if **f** was a proposition, then its meaning is dependent on the meaning of negation, and conversely, the meaning of negation is dependent on the meaning of **f**: their respective meanings can only be given together. Such a circular dependence of meaning might be philosophically objectionable. On the other hand, it might be worth noting that these rules only give rise to a very weak negation, as **f** could be interpreted as any unacceptable sentence (*cf.* the negation of minimal logic). What makes $\bot$ a good candidate for a definition of negation is that it entails everything, but this may not be the case for **f** if Thinning is not a structural rule for the salient grouping of Θ. Thus a certain amount

[18] Neil Tennant suggests to liken a symbol like **f** to an interjection 'Contradiction!': it marks a 'dead end' in a deduction, see his 'Negation, Absurdity and Contrariety' in *What is Negation?* edd. Dov Gabbay and Heinrich Wansing (Dortrecht: Kluwer, 1999) 199-222

of stipulation is needed here, at least where relevance logics are concerned. But this is not the place to get into this in any further detail.

The two negation rules only give negations close to negations close to intuitionist negation. To get negations close to classical negation, we need look at the rule for ¬E in a slightly deviant way. Suppose this is taken to be, not an elimination rule for $\neg A$, but as an elimination rule for A.[19] Strictly speaking, the order of the premises then has to be changed, but this makes it an elimination rule almost of form one:

$$\frac{Z \vdash A \qquad Y \vdash \neg A}{\Theta(Z\ Y) \vdash \mathbf{f}}$$

This rule is of course redundant if the above rule of ¬E is present in the logic. But it shows that the corresponding introduction rule for A which corresponds to this elimination rule for A is *consequentia mirabilis*:

$$\frac{\Theta(X\ \neg A) \vdash \mathbf{f}}{X \vdash A}$$

Conversely, considering *consequentia mirabilis* to be an introduction rule for A makes it a rule almost of form one, the corresponding elimination rule of which is ¬E with the order of premises changed.

Hence, lifting the restriction that only one constant figures in a rule, we can say that negations are governed by rules almost of form one, where negations close to intuitionist negation are governed by ¬I and ¬E, and, reading *consequentia mirabilis* in a slightly deviant way, we can say that negations related to classical negation are also governed by rules almost of

[19]This suggestion has been made by Peter Milne in 'Classical Harmony: Rules of Inference and the Meanings of the Logical Constants', *Synthese* 100 (1994), 49-94, p.58

form one, i.e. ¬I, ¬E and *consequentia mirabilis* (the version of ¬E with the order of premises reversed being redundant). Thus classical negation is no worse from a perspective of form than intuitionist negation and we can characterise them both as governed by harmonious rules, as defined in the next section.

Thus, from a formal perspective, the rules of classical negation exhibit the same kind of symmetry as those of intuitionist negation. To show that this symmetry is also philosophically adequate, we need to ensure that *consequentia mirabilis* does not upset the equilibrium between grounds for asserting complex formulas and their consequences: *consequentia mirabilis* introduces potentially new grounds for asserting A, hence an application of this rule followed by an application of an elimination rule for the main connected of A counts as a local peak. So we need to show that these local peaks can be levelled.

This leads to a somewhat puzzling result in the philosophical interpretation: as local peaks with implication and *consequentia mirabilis* can be levelled in classical logic, *consequentia mirabilis* does not introduce new grounds for asserting formulas of the form $A \supset B$. Nonetheless, there are theorems containing $\supset$ but not containing negation, in particular $((A \supset B) \supset A) \supset A$, the proof of which needs to appeal to *consequentia mirabilis*. But classical negation does not need $\supset$ as a primitive, as it could be defined in terms of ¬ and &, for which no corresponding problem arises, so I don't think this matters much.

The procedure of viewing *consequentia mirabilis* as an introduction rule for A only works because of the presence of **f**. Suppose we viewed, e.g., →E to be an introduction rule for B:

$$\frac{X \vdash A \to B \qquad X \vdash A}{X \vdash B}$$

If we pretended that this is a rule of form one, it would determine two elimination rules for B, namely:

$$\frac{X \vdash B}{X \vdash A \to B} \qquad \frac{X \vdash B}{X \vdash A}$$

The second rule is obviously problematic. It forces us to interpret B as equivalent to $\bot$, i.e. not as an arbitrary proposition, but as one with a specific interpretation.

2.3 Harmony, Stability and Normality

We are now in a position to give a formally precise definition of harmony: introduction and elimination rules for a constant Ξ are *in harmony* or a logical constant Ξ is governed by *harmonious* rules iff either (i) Ξ is governed by an introduction rule of type one and elimination rules read off it by the method in section 2.2.2, or (ii) Ξ is governed by an elimination rule of type two and introduction rules read off it by the method in section 2.2.3, or (iii) Ξ is governed by negation rules as in section 2.2.4.

Stability is sometimes characterised as harmony plus its converse. Given that the process of reading off introduction and elimination rules from each other can be inverted, we would now be faced with the consequence that stability would be a notion that adds nothing to harmony. Thus I suggest to use a different notion of stability.

The definition of harmony says nothing about conditions that might be imposed on the application of rules of inference, as in quantum disjunction or **S4** necessity discussed earlier. These rules are of the form of type two and type one, respectively. The rules for quantum disjunction have the same form as the rules for intuitionist disjunction, but there is a restriction on the application of the rule, which requires that in the collateral deductions,

$Y(A) = A$ and $Y(B) = B$. The rules for **S4** necessity have the same form as the rules for the truth-operator:

$$\frac{X \vdash A}{X \vdash \mathbf{T}A} \qquad \frac{X \vdash \mathbf{T}A}{X \vdash A}$$

The difference is that the application of $\Box$I is restricted to cases where all formulas on X are of the form $\Box B$. The restrictions do not affect the forms of the rules, they only restrict their application. Hence the rules for quantum disjunction and necessities are harmonious according to my definition.

The definition of harmony just given allows for restrictions on the applications of the rules. As discussed earlier, Dummett would not count the rules for quantum disjunction as stable. This suggests the following, non-trivial definition of stability: introduction and elimination rules for a constant Ξ are *stable* or a logical constant Ξ is governed by *stable* rules iff the rules for Ξ are harmonious there are no restrictions on the application of the rules. Hence $\veebar$ and $\Box$ are governed by harmonious, but not stable rules, according to my definitions.

My definitions of harmony and stability are purely formalistic and apply to rules of inference in isolation of logics they occur in. Harmony and stability normally do not guarantee normality, i.e. that the deductions in the logic normalise, which is the philosophically crucial notion in the context of a justification of deduction. To establish that there are reduction procedures for removing local peaks and local maxima from deductions depends on the logics in questions. Reduction procedures will be given in the next section for the case where there are no applications of structural rules in the local peaks and maxima, *i.e.* for direct local peaks and maxima. A step in establishing that a logic normalises thus consists in showing that stretched local peaks and maxima may be transformed into direct ones, which is the first Lemma to be proved in section 2.5.2.

The reduction procedures can also be conceived of merely formalistically as methods for transforming certain trees into other trees. Normalisation of deductions, however, is a process rather more subtle than that. It concerns particular deductions of particular formal systems, and the question is how far it may be generalised. This question is answered by the reduction procedures to be given in the next section to the extent that they exhibit the general patterns of steps in the normalisation of deductions. These patterns are indeed always the same. But whether they are applicable in a proof that shows that the deductions of a specific logic normalise depends on whether the rules of the system are such that it is guaranteed that applying the reduction procedures always transforms deductions into deductions. This has to be established for each logic individually; if it is the case, the deductions normalise and it is a justified logic. There is a class of logics where it is particularly easy to establish that the rules of the system have the desired property: these are the logics which have only stable rules for connectives. In such a case it suffices to inspect the rules in order to know that the deductions in the logic normalise: that this is so is guaranteed by the very form of the rules. If some rules of a logic are merely harmonious and involve restrictions on applications of rules, whether deductions normalise depends also on other factors, namely the nature of the restrictions and the other rules present in the system.

Despite their philosophical importance deriving from the view that stable rules of inference completely determine the meaning of the constant they govern, from the perspective of normalisation there is not much of a difference between logics which have only stable rules and logics where the reduction procedures are applicable, even though some of the rules are only harmonious. We may call a logic L *regular* if its rules ensure that the reduction procedures always turn deductions of L into deductions of L and its empty list is genuine.

2.4 Reduction Procedures

One minor obstacle to reduction procedures turning deductions of a logic into deductions of that logic are the restrictions on parameters in the rules. In unfortunate circumstances it could happen that after a reduction procedure has been applied, an application of such a rule fails to be correct, as the restriction on the parameters are unfulfilled. As there is an unlimited amount of parameters at our disposal in the language for each logic, it is always possible to rewrite a deduction in such a way that the variables on which the restrictions are imposed do not occur in any other subdeduction of the deduction than that one which leads to the conclusion of the application of the rule. This is fairly obvious, but the proof is tedious, so I won't give it here. In the following, I'll always assume the necessary changes of parameters to have been made to avoid conflicts in the restrictions.

2.4.1 Reduction Procedures for Maximal Formulas

A *maximal string* in a deduction is a string consisting of occurrences of a formula A which is conclusion of an introduction rule for its main connective Ξ, of *ex falso quodlibet* or of *consequentia mirabilis* and major premise of an elimination rule of Ξ. A *maximal formula* is a maximal string with exactly one formula on it. A *local peak with* Ξ consists of the maximal string, the premises of the introduction rule of which it is the conclusion and the minor premises and conclusion of the elimination rule of which it is the major premise; *i.e.* the local peak consists of the context in which the maximal string occurs in a deduction. If the maximal string of the local peak consists of only one formula, the local peak may be called *direct*, *stretched* otherwise.

For Rules of Type One. In a local peak with a constant Ξ governed by rules of type one, an application of its introduction rule is followed by an application of one of its elimination rules ΞE_o:

$$
\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Phi(X\ A_1\dots A_h)\vdash B_1\end{matrix} \quad \dots \quad \begin{matrix}\Pi_o\\ \Theta(X\ A_i\dots A_j)\vdash B_o\end{matrix} \quad \dots \quad \begin{matrix}\Pi_p\\ \Psi(X\ A_k\dots A_l)\vdash B_p\end{matrix}}{X\vdash \Xi\,\overline{x}\,A_1\dots A_k B_1\dots B_p[\overline{a}/\overline{x}]} \quad \begin{matrix}\Sigma_i\\ Y_i\vdash A_i[\overline{a}/\overline{t}]\end{matrix} \quad \dots \quad \begin{matrix}\Sigma_j\\ Y_j\vdash A_j[\overline{a}/\overline{t}]\end{matrix}}{\begin{matrix}\Theta(X\ Y_i\dots Y_j)\vdash B_o[\overline{a}/\overline{t}]\\ \mathrm{P}\end{matrix}}
$$

The local peak may be levelled and the maximal formula $\Xi\,\overline{x}\,A_1\dots A_j B_1\dots B_p[\overline{a}/\overline{x}]$ removed by re-organizing the deduction with the following method. If $A_i\dots A_j$ are replaced by $Y_i\dots Y_j$, respectively, on the branches beginning with these formulas in the bottom-most consecution $\Theta(X\ A_i\dots A_j)\vdash B_o$ of Π_o and $\overline{a}$ is replaced by $\overline{t}$ throughout Π_o, then the initial consecutions in which the branches end will be $Y_i\vdash A_i[\overline{a}/\overline{t}]\ \dots\ Y_j\vdash A_j[\overline{a}/\overline{t}]$. Let's denote the result of this procedure by $\Pi_o[A_i/Y_i\dots A_j/Y_j][\overline{a}/\overline{t}]$. Notice that square brackets containing lists work rather differently from squarebrackets containing terms: the latter denote replacement throughout a formula/list/deduction, the former only on previously specified branches. To construct the desired deduction of $\Theta(X\ Y_i\dots Y_j)\vdash B_o[\overline{a}/\overline{t}]$ without the local peak, we append $\Sigma_i\dots\Sigma_j$ to those top-nodes of $\Pi_o[A_i/Y_i\dots A_j/Y_j][\overline{a}/\overline{t}]$ which begin with the consecutions $Y_i\vdash A_i[\overline{a}/\overline{t}]\ \dots\ Y_j\vdash A_j[\overline{a}/\overline{t}]$ resulting from the replacement. Using the notation introduced in section 1.3.3, the re-organised deduction is:

$$
\begin{array}{ccc}
\Sigma_i & & \Sigma_j \\
Y_i \vdash A_i[\overline{a}/\overline{t}] & \ldots & Y_j \vdash A_j[\overline{a}/\overline{t}] \\
\hline\hline
\multicolumn{3}{c}{\Pi_o[A_i/Y_i \ldots A_j/Y_j][\overline{a}/\,\overline{t}\,]} \\
\multicolumn{3}{c}{\Theta(X\ Y_i \ldots Y_j) \vdash B_o[\overline{a}/\overline{t}]} \\
\multicolumn{3}{c}{\mathrm{P}}
\end{array}
$$

Implementing the reduction procedure might produce new maximal formulas. First, this may happen if B_o in Π_o is a conclusion of an introduction rule for its main premise and $B_o[\overline{a}/\overline{t}]$ in P is major premise of an elimination rule, and secondly if for instance A_i in an initial consecution of Π_o is major premise of an elimination rule and the last step of Σ_i is an introduction rule for its main connective. Evidently, any new maximal formula is of *lower degree* than the maximal formula that has been removed, *i.e.* less constants occur in it, as any new maximal formula is a subformula of the removed one, and the resulting deduction also contains *less* applications of rules of inference. This yields a suitable bases for a proof by induction that all maximal formulas can be removed.

For Rules of Type Two In a local peak with a constant Ξ governed by rules of type two, an application of one of its introduction rules ΞI_o is followed by an application of its elimination rule:

$$
\dfrac{\dfrac{\begin{array}{c}\Pi_j\\ X_j \vdash D_j[\overline{x}\,/\,\bar{t}\,]\end{array} \quad \dots \quad \begin{array}{c}\Pi_k\\ X_k \vdash D_k[\overline{x}\,/\,\bar{t}\,]\end{array}}{\Theta(X_j \dots X_k) \vdash \Xi\,\overline{x}\,D_1 \dots D_n} \quad \begin{array}{c}\Sigma_1\\ Y(\Phi(D_1[\overline{x}/\overline{a}] \dots D_i[\overline{x}/\overline{a}])) \vdash E\end{array} \quad \dots \quad \begin{array}{c}\Sigma_p\\ Y(\Psi(D_l[\overline{x}/\overline{a}] \dots D_n[\overline{x}/\overline{a}])) \vdash E\end{array}}{\begin{array}{c}Y(\Theta(X_j \dots X_k)) \vdash E\\ \mathrm{P}\end{array}}
$$

It is understood that there is a deduction Σ_o of $\Theta(D_j[\overline{x}/\overline{a}] \dots D_k[\overline{x}/\overline{a}]) \vdash E$ amongst the collateral deductions of minor premises, which I couldn't add explicitly due to limitations of space on the page.

The local peak may be removed by a method similar to the one given for rules of type one: the branches starting with $D_j[\overline{x}/\overline{a}] \dots D_k[\overline{x}/\overline{a}]$ in the bottom-most consecution $\Theta(D_j[\overline{x}/\overline{a}] \dots D_k[\overline{x}/\overline{a}]) \vdash E$ of Σ_o are replaced by $X_j \dots X_k$, $\overline{a}$ is replaced by $\bar{t}$, and $\Pi_j \dots \Pi_k$ are appended to the consecutions in which the branches end. More graphically, the local peak in a deduction is replaced by the following construction:

$$
\begin{array}{c}
\begin{array}{c}\Pi_j\\ X_j \vdash D_j[\overline{x}\,/\,\bar{t}\,]\end{array} \quad \dots \quad \begin{array}{c}\Pi_k\\ X_k \vdash D_k[\overline{x}\,/\,\bar{t}\,]\end{array}\\
= =\\
\Sigma_o[D_j/X_i \dots D_k/X_k][\overline{a}/\bar{t}]\\
\Theta(X_j \dots X_k) \vdash E\\
\mathrm{P}
\end{array}
$$

As in the previous case, applying the reduction procedure might introduce new maximal formulas, namely if, *e.g.*, D_j in an initial consecution of Π_j is major premise of an elimination rule and $D_j[\overline{a}/\overline{t}]$ is conclusion of an introduction rule in Σ_o, or if E is conclusion of an introduction rule in the last step of Σ_o and major premise of an elimination rule in P. In the previous case, the new maximal formula has less connectives than the removed one. But now there is no guarantee that the new maximal formula is of lower degree than the removed one, but the resulting deduction contains less applications of rules of inference than the original deduction. What is more, the reduction procedures given in section 2.4.2 and Lemma 3 to be proved in section 2.5.2 ensures that deductions can be transformed in such a way that no conclusion of a rule of type two is major premises of an elimination rule, hence cannot become a maximal formula by applying the reduction procedure.

Reduction Procedures for *ex falso quodlibet* An application of *ex falso quodlibet* followed by an application of an elimination rule with the formula so derived as its major premise constitutes a local peak, which should be removable from deductions. Special reduction procedures for this case are not necessary, as the reduction procedures of section 2.4.2 and Lemma 3 take care of this as a special case.

Such local peak could also never arise if the conclusion of *ex falso quodlibet* is required to be atomic. In section 2.5.10, I'll consider under which conditions it is possible to impose this restriction.

2.4.2 Reduction Procedures for Maximal Segments

A *maximal segment (of minor premises)* is a sequence of strings $\sigma_1 \ldots \sigma_n$ such that σ_1 is the conclusion of an introduction rule for the principal operator of the formula on it or of *ex falso quodlibet*, σ_n is main premise of an elimination rule, and for each i, j, $1 \leq i < j < n$, σ_i is minor premise of an application of a rule of form two and σ_j is the conclusion of this rule. It follows that the formulas on the strings $\sigma_1 \ldots \sigma_n$ are all of the same shape. We may call it a *maximal minor premise.* The part of a deduction in which a maximal segment occurs, *i.e.* the segment plus premises and conclusions of rules of inference applied to its formulas, may be called a *local maximum.*

If the last string of the segment consists of only one occurrence of the formula, the local maximum may be called *direct*, *stretched* otherwise. Reduction procedures will be given for direct local maxima, *i.e.* to prove normalisation a lemma is needed that establishes that every stretched local maximum may be turned into a direct one. Let the *weight* of a maximal segment be the number of maximal minor premises on its strings. In the normalisation proof to be given in section 2.5.2, it is shown by induction over the weight of maximal segments that any segment can be reduced to weight 1, in which case it turns into a maximal formula, which can be removed by the reduction procedures of the last section.

The two reduction procedures given in the next section apply more generally to the case where the conclusion of an elimination rule of type two is the major premises of an elimination rule; maximal segments are a special case thereof.

Reduction Procedure One Suppose the conclusion of an application of an elimination rule ΞE of form two with major premise $\Xi\,\overline{x}\,D_1 \ldots D_n$ and minor premises of the form $\mathrm{O}\,\overline{x}\,A_1 \ldots A_l B_1 \ldots B_p$, where O is governed by a rule of form one, is followed by an application of $\mathrm{O}E_o$:

$$\dfrac{\dfrac{Z \vdash \Xi\,\overline{y}\,D_1 \ldots D_n \qquad Y(\Phi(D_1[\overline{y}/\overline{a}] \ldots D_i[\overline{y}/\overline{a}])) \vdash \mathrm{O}\,\overline{x}\,A_1 \ldots A_l B_1 \ldots B_p \quad \ldots \quad Y(\Psi(D_l[\overline{y}/\overline{a}] \ldots D_n[\overline{y}/\overline{a}])) \vdash \mathrm{O}\,\overline{x}\,A_1 \ldots A_l B_1 \ldots B_p}{Y(Z) \vdash \mathrm{O}\,\overline{x}\,A_1 \ldots A_l B_1 \ldots B_p} \qquad X_i \vdash A_i[\overline{x}/\overline{t}] \quad \ldots \quad X_j \vdash A_j[\overline{x}/\overline{t}]}{\Theta(Y(Z)\ X_i \ldots X_j) \vdash B_o[\overline{x}/\overline{t}]}$$

The application of $\mathrm{O}E_o$ can be *pushed up* one step, to *before* the application of ΞE, so that the minor premises of the latter rule have the form $B_o[\overline{x}/\overline{t}]$. Then instead of applying $\mathrm{O}E_o$ to the consecutions $Y(Z) \vdash \mathrm{O}\,\overline{x}\,A_1 \ldots A_k B_1 \ldots B_p$ and $X_i \vdash A_i[\overline{x}/\overline{t}]$ $\ldots$ $X_j \vdash A_j[\overline{x}/\overline{t}]$, it is applied to the consecutions that have the minor premises of ΞE in the succedents as major premises and $X_i \vdash A_i[\overline{x}/\overline{t}]$ $\ldots$ $X_j \vdash A_j[\overline{x}/\overline{t}]$ as minor premises. So for the first minor premise of ΞE, this yields the following application of $\mathrm{O}E_o$:

$$\dfrac{Y(\Phi(D_1[\overline{y}/\overline{a}] \ldots D_i[\overline{y}/\overline{a}])) \vdash \mathrm{O}\,\overline{x}\,A_1 \ldots A_l B_1 \ldots B_p \qquad X_i \vdash A_i[\overline{x}/\overline{t}] \qquad \ldots \qquad X_j \vdash A_j[\overline{x}/\overline{t}]}{\Theta(Y(\Phi(D_1[\overline{y}/\overline{a}] \ldots D_i[\overline{y}/\overline{a}]))\ X_i \ldots X_j) \vdash B_o[\overline{x}/\overline{t}]}$$

If this is done for all minor premises of ΞE, this rule may be applied thus:

$$\dfrac{Z \vdash \Xi\,\overline{y}\,D_1 \ldots D_n \qquad \Theta(Y(\Phi(D_1[\overline{y}/\overline{a}] \ldots D_i[\overline{y}/\overline{a}]))\ X_i \ldots X_j) \vdash B_o[\overline{x}/\overline{t}] \qquad \ldots \qquad \Psi(Y(\Phi(D_l[\overline{y}/\overline{a}] \ldots D_n[\overline{y}/\overline{a}]))\ X_i \ldots X_j) \vdash B_o[\overline{x}/\overline{t}]}{\Theta(Y(Z)\ X_i \ldots X_j) \vdash B_o[\overline{x}/\overline{t}]}$$

If $\mathrm{O}\,\overline{x}\,A_1...A_l B_1...B_p$ is the last formula of a maximal segment, the weight of the local maximum is thereby reduced by one. Note that the procedure does not create new maximal segments or maximal formulas and neither does it increase the length of any other maximal segment existing in the deduction: the only candidate here would be B_o, but it is the conclusion of ΞE.

Reduction Procedure Two Suppose the conclusion of an application of an elimination rule ΞE of form two with major premise $\Xi\,\overline{x}\,D_1 \ldots D_n$ and minor premises of the form $\mathrm{O}\,\overline{x}\,C_1 \ldots C_q$ followed by an application of $\mathrm{O}E$ also of form two:

$$\dfrac{\dfrac{Z \vdash \Xi\,\overline{y}\,D_1...D_n \qquad Y(\Phi(D_1[\overline{y}/\overline{a}]...D_i[\overline{y}/\overline{a}])) \vdash \mathrm{O}\,\overline{x}\,C_1...C_p \quad ... \quad Y(\Psi(D_l[\overline{y}/\overline{a}]...D_n[\overline{y}/\overline{a}])) \vdash \mathrm{O}\,\overline{x}\,C_1...C_p}{Y(Z) \vdash \mathrm{O}\,\overline{x}\,C_1...C_p} \qquad V(\Theta(C_1[\overline{x}/\overline{b}]...C_h[\overline{x}/\overline{b}]) \vdash E \quad ... \quad V(\Omega(C_k[\overline{x}/\overline{b}]...C_p[\overline{x}/\overline{b}]) \vdash E}{V(Y(Z)) \vdash E}$$

The application of $\mathrm{O}E$ can be *pushed up* one step to *before* the application of ΞE, so that the minor premises of the latter rule are occurrences of the formula E. So instead of applying $\mathrm{O}E$ to $Y(Z) \vdash \mathrm{O}\,\overline{x}\,C_1 \ldots C_p$ and $V(\Theta(C_1[\overline{x}/\overline{b}] \ldots C_h[\overline{x}/\overline{b}]) \vdash E$ $\ldots$ $V(\Omega_{\mathrm{O}}(C_k[\overline{x}/\overline{b}] \ldots C_p[\overline{x}/\overline{b}]) \vdash E$, it is applied to the consecutions that have the minor premises of ΞE in their succedents as major premises and those that have E in their succedents as minor premises. For the first minor premise of ΞE, the application of $\mathrm{O}E$ then is:

$$\dfrac{Y(\Phi(D_1[\overline{y}/\overline{a}] \ldots D_i[\overline{y}/\overline{a}])) \vdash \mathrm{O}\,\overline{x}\,C_1 \ldots C_p \qquad V(\Theta(C_1[\overline{x}/\overline{b}] \ldots C_h[\overline{x}/\overline{b}]) \vdash E \qquad \ldots \qquad V(\Omega(C_k[\overline{x}/\overline{b}] \ldots C_p[\overline{x}/\overline{b}]) \vdash E}{V(Y(\Phi(D_1[\overline{y}/\overline{a}] \ldots D_i[\overline{y}/\overline{a}]))) \vdash E}$$

If this is done for all the minor premises of ΞE, this rule may be applied thus:

$$\frac{Z \vdash \Xi\,\overline{y}\,D_1 \ldots D_n \qquad V(Y(\Phi(D_1[\overline{y}/\overline{a}] \ldots D_i[\overline{y}/\overline{a}]))) \vdash E \qquad \ldots \qquad V(Y(\Psi(D_1[\overline{y}/\overline{a}] \ldots D_i[\overline{y}/\overline{a}]))) \vdash E}{V(Y(Z)) \vdash E}$$

If $\mathrm{O}\,\overline{x}\,C_1 \ldots C_p$ is the last formula of a maximal segment, the weight of the local maximum is thereby reduced by one. Note that this time although the number of maximal segments of formulas cannot be increased by applying the method, the *length* of a maximal segment might be, namely if E in the original deduction was part of one. This will have to be taken care of in the normalisation proof.

2.5 Quasi-Intuitionist Logics

Call a logic *quasi-intuitionist* if it is regular and has only rules of inference of forms one and two, but no negations as in section 2.2.4, i.e. negation is defined as in terms of $\bot$ and implication. These are the logics to be considered in this section.

2.5.1 Uniqueness

Connectives governed by rules of type one or type two *unique* in the following sense:

Theorem 1 (Uniqueness)**.** If two of its connectives Π and Ξ are governed by the same rules of inference of form one or two, then $\Pi\ \overline{x}\ A_1 \dots A_n \dashv\vdash \Xi\ \overline{x}\ A_1 \dots A_n$.

Proof. For rules of type one, to show that $\Pi\overline{x}A_1 \dots A_n \vdash \Xi\overline{x}A_1 \dots A_n$, apply each of the p elimination rules for Π to the initial consecution $\Pi\overline{x}A_1 \dots A_n \vdash \Pi\overline{x}A_1 \dots A_n$ and whichever of $A_1 \vdash A_1 \dots A_n \vdash A_n$ are necessary. Then apply ΞI. The converse is similar, and so is the proof in case of rules of type two. *Q.e.d.*

2.5.2 Normalisation

The reduction procedures of section 2.4.1 only apply to direct local peaks, where the introduction rule is followed immediately by the elimination rule, but not for stretched ones. Analogously for the reduction procedures of section 2.4.2. As the lists on which major premises of elimination rules depend contain no discharged assumptions, they are carried down unchanged to be part of the list on which the conclusion of the rule depends. Thus any application of a structural rule to the list on which the major premise depends may be moved downwards to after the application of the rule. It follows that every stretched local peak/maximum can be transformed into a direct one:

Lemma 2. Any deduction may be transformed into one where the major premises of elimination rules are not conclusions of structural rules.

Proof. Suppose the main premise of an elimination rule of type one is premise of a structural rule $Z \Leftarrow Z'$:

$$\dfrac{\dfrac{Z \vdash \Xi\,\overline{x}\,A_1 \dots A_k B_1 \dots B_p}{Z' \vdash \Xi\,\overline{x}\,A_1 \dots A_k B_1 \dots B_p} \qquad Y_i \vdash A_i[\overline{x}/\overline{t}] \qquad \dots \qquad Y_j \vdash A_j[\overline{x}/\overline{t}]}{\Theta(Z'\ Y_i \dots Y_j) \vdash B_o[\overline{x}/\overline{t}]}$$

The application of the structural rule can be moved down to after the application of the elimination rule, by replacing the above by the following construction shows:

$$\dfrac{\dfrac{Z \vdash \Xi\,\overline{x}\,A_1 \dots A_k B_1 \dots B_p \qquad Y_i \vdash A_i[\overline{x}/\overline{t}] \qquad \dots \qquad Y_j \vdash A_j[\overline{x}/\overline{t}]}{\Theta(Z\ Y_i \dots Y_j) \vdash B_o[\overline{x}/\overline{t}]}}{\Theta(Z'\ Y_i \dots Y_j) \vdash B_o[\overline{x}/\overline{t}]}$$

Similarly for rules of type two. An application of the structural rule:

$$\dfrac{\dfrac{Z \vdash \Xi\,\overline{x}\,D_1 \dots D_n}{Z' \vdash \Xi\,\overline{x}\,D_1 \dots D_n} \quad Y(\Phi(D_h[\overline{x}/\overline{a}] \dots D_i[\overline{x}/\overline{a}])) \vdash E \quad \dots \quad Y(\Psi(D_l[\overline{x}/\overline{a}] \dots D_m[\overline{x}/\overline{a}])) \vdash E}{Y(Z') \vdash E}$$

can be moved down one step:

$$\dfrac{\dfrac{Z \vdash \Xi\,\overline{x}\,D_1 \dots D_n \quad Y(\Phi(D_h[\overline{x}/\overline{a}] \dots D_i[\overline{x}/\overline{a}])) \vdash E \quad \dots \quad Y(\Psi(D_l[\overline{x}/\overline{a}] \dots D_m[\overline{x}/\overline{a}])) \vdash E}{Y(Z) \vdash E}}{Y(Z') \vdash E}$$

What is needed is a systematic way of applying the transformation methods to a deduction to insure that after a finite number of steps no major premises of elimination rules are conclusions of applications of structural rules. One option is to apply the method to the leftmost major premise of an elimination rule which is a conclusion of a structural rule so that no other major premise of an elimination rule in the subdeduction leading to it is also a conclusion of a structural rule. Applying the procedure a finite number of times leads to the desired result. *Q.e.d.*

The reduction procedures of section 2.4.2 may be used to prove something slightly stronger than that maximal segments may be removed:

Lemma 3. Any deduction can be transformed into a deduction in which the strings beginning with the conclusion of elimination rules of type two are not major premises of elimination rules.

Proof. The lemma is proved by alternating applications of lemma 2 and the reduction procedures of section 2.4.2. As applying the former may increase the length of segments of minor premises, a systematic application is required to ensure that after a finite number of steps all formulas which are conclusion of an elimination rule of type two and major premise of an introduction rule are removed. The consecutions in deductions may in an obvious way be assigned a *level* in the deduction: the conclusion is on level 0, its premises on level 1, their premise on level 2, and so on. Applying first Lemma 2 and then the reduction procedures to a segment the last string of which is major premise of an elimination rule and the last formula of which is of lowest level cannot increase the length of any maximal segment. There may be more than one such segment, so in order to get a systematic way of applying the method, choose the leftmost one to start with. The Lemma may then be proved by induction over the sum of the weights of maximal segments. *Q.e.d.*

Applying lemma 3 transforms any local maximum into a local peak.

Applying the reduction procedures for local peaks may produce new local peaks, so it needs to be ensured that the normalisation procedure comes to an end somewhere. If maximal segments have been removed from a deduction, new local peaks that may arise from application of the reduction procedures for maximal formulas are such that their maximal formula are of a degree less than the one of the peak that has been removed. Let the *burden* ω of a deduction be the sum of the degrees of its maximal formulae of highest degree. The normalisation theorem is proved by induction over this number.

Theorem 4 (Normalisation)**.** For every deduction Π such that $\Pi(X \vdash_{\mathrm{L}} A)$, there is a deduction Π' in normal form such that $\Pi'(X \vdash_{\mathrm{L}} A)$.

Proof. By lemmata 2 and 3, Π may be transformed into a deduction where every stretched local peak has been transformed into a direct one and every local maximum has been reduced to a local peak. Call the resulting deduction Π^*. It remains to apply the reduction procedures for local peaks to Π^* in such a way as to ensure that all maximal formulas are removed. Take the leftmost maximal formula of lowest level and apply the reduction procedure. This lowers the burden of the deduction, as in Π^* there are no conclusions of elimination rules of type 2, which are premises of elimination rules, and so by induction the theorem holds. *Q.e.d.*

2.5.3 The Form of Proofs

A is a theorem of logic L iff there is a deduction Π of L such that $\Pi(0 \vdash A)$. To formulate a provable version of the fundamental 'assumption' care needs to be taken of the possibility that a deduction ends with steps of applications of structural rules which reduce a list with only a number of 0s on it to 0.

Recall that empty lists are required to be genuine in regular logics. Call a list on which only 0s occur a *null-list*. If 0 is genuine in L, for every null-list $\Theta(0 \ldots 0)$ there is what may be called a *deconstruction* that shows that $\Theta(0 \ldots 0) \Leftarrow 0$ is a derived structural rule of L. Then we may say that a proof *terminates* in an application of an operational rule if at most a deconstruction of a null-list follows this application.

Theorem 5. Let Π be a proof of A in normal form. Then Π terminates in an application of an introduction rule for the main connective of A.

Proof. The proof is by induction over the number k of applications of operational rules in Π.

In case there is only one application of an operational rule in Π, the rule applied cannot be an elimination rule of type 1 nor an introduction rule of type 2, as they do not discharge assumptions. Furthermore, the conclusion of a single application of an elimination rule of type 2 always depends on at least the major premise, which excludes this case as well. Hence, if only one rule is applied in Π, it can only be an introduction rule of type 1, with X an empty list and all the premises are also discharged assumptions (*i.e.* the rule is applied to initial consecutions). But then the deduction terminates in an application of an introduction rule for the main connective of A, which completes the basis of the induction.

To prove the induction step, four cases are to be considered. Cases 1 and 2 are trivial: the last application of an operational rule in Π is an application of an introduction rule of type 1 or 2. Then its conclusion is $Z \vdash A$, where Z is a null-list. In other words, the proof terminates in an application of an introduction rule for the main connective of A.

Case 3. Suppose the last operational rule applied in Π is an elimination rule of type 1 with the major premise $\Xi\overline{x}A_1 \ldots A_n B_1 \ldots B_p$. As Π is a proof, $\Xi\overline{x}A_1 \ldots A_n B_1 \ldots B_p$ must depend on a null-list. As Π is in normal form, the sub-deduction leading to $\Xi\overline{x}A_1 \ldots A_n B_1 \ldots B_p$ is obviously in normal form, too. But by induction hypothesis, it terminates in an application of an introduction rule. So $\Xi\overline{x}A_1 \ldots A_n B_1 \ldots B_p$ is the maximal formula of a stretched local peak, contrary to hypothesis that Π is in normal form. Hence Π cannot end in an application of an elimination rule of type 1.

Case 4. Suppose the last operational rule applied in Π is an elimination rule of type 2 with the major premise $\Xi\overline{x}D_1 \ldots D_n$. As in Case 3, as Π is a proof, $\Xi\overline{x}D_1 \ldots D_n$ must depend on a null-list. As Π is in normal form, the sub-deduction leading to $\Xi\overline{x}D_1 \ldots D_n$ is obviously in normal form, too. But by induction hypothesis, it terminates in an application of an introduction rule. So $\Xi\overline{x}D_1 \ldots D_n$ is the maximal formula of a stretched local peak, contrary to hypothesis that Π is in normal form. Hence Π cannot end in an application of an elimination rule of type 2.

Therefore the only possible cases are case 1 and 2, which completes the induction step. *Q.e.d.*

Corollary 6. If there is a proof of A, then there is a proof that terminates in an application of an introduction rule for the main connective of A.

Proof. Let Π be a proof of A. By theorem 4 there is a proof Π' of A in normal form which, by theorem 5 terminates in an application of an introduction rule of the main connective of A.

2.5.4 Consistency

The consistency of quasi-intuitionist logics follows in the form of two corollaries:

Corollary 7 (Consistency 1)**.** There is no proof of $\bot$.

Proof. By corollary 6, if there is a proof of a formula A, then there is a proof that terminates in an application of the introduction rule for its main connective. But $\bot$ does not have an introduction rule. Hence there is no proof of $\bot$.

Corollary 8 (Consistency 2)**.** There is no proof of the atomic formula p.

Proof. Suppose Π is a proof of p. Then by corollary 6 there is a proof Π' of p that terminates with an application of the introduction rule for the main connective of p. But p does not have a main connective. Hence Π is not a proof.

It also follows that $\neg p$, *i.e.* $p \rightarrow \bot$, is not provable either, for any implication $\rightarrow$. For if it was, we could transform its proof into a proof of $A \rightarrow \bot$, for some theorem A of the logic, by replacing the relevant occurances of p in its proof by A, so that, by applying the elimination rule for $\rightarrow$, $\bot$ would be provable, contradicting Corollary 7.

2.5.5 Paths in Deductions of Normal Form

The next step is to prove that connectives of type one and two can be added conservatively to quasi-intuitionist logics. This is shown in section 2.5.7, but some preparations are needed, to provide which is the purpose of sections 2.5.5 and 2.5.6. I'm leaning heavily on Prawitz' treatment of the topic in *Natural Deduction*, Chapters II and III. The theorems given here are generalisations of his results to any quasi-intuitionist logic.

Recall that we may speak of strings of formulas being the conclusions and premises of applications of rules of inference. If an application of a rule discharges an assumption in the antecedent of the consecution containing a minor premise, then there is a branch beginning with this formula and ending either in antecedents of an initial consecution or in formulas introduced into antecedents by Thinning; call the formulas in the succedents of the initial consecutions the *ancestors* of the discharged assumptions.

Call a *path* π in a deduction a sequence of strings $\varsigma_1 \ldots \varsigma_n$ such that

(i) the formula on ς_1 is a formula introduced into the deduction as the succedent of an initial consecution, but is not an ancestor of an assumption discharged by an application of a rule of type 2,

(ii) ς_n is either the conclusion of the deduction or a minor premise of an application of a rule of type 1 or a major premise $\Xi\overline{x}D_1 \ldots D_n$ of an application of an elimination rule of type 2, if assumption D_j discharged by it has no ancestors (*i.e.* is introduced by Thinning),

(iii) for $i < n$, if ς_i is major premise of an application of a rule of type 2 which discharges assumptions, then ς_{i+1} is an ancestor of an assumption discharged by it, and if ς_i is premise of a rule not mentioned so far (note that this includes the *falsum* rule), then ς_{i+1} is the conclusion of the rule.

Explanation. There are n paths through a major premise $\Xi\overline{x}D_1 \ldots D_n$ of an application of an elimination rule of type 2, one for each of its immediate subformulas D_j, which end in $\Xi\overline{x}D_1 \ldots D_n$ if the formula discharged by an application of the rule and corresponding to the subformula D_j has no ancestors.

We may say that the strings $\varsigma_1 \ldots \varsigma_n$ are *on* the path.

The notion of a maximal segment is easily generalised to cover sequences of strings which aren't necessarily maximal: call a *segment* σ a sequence of strings $\varsigma_1 \ldots \varsigma_n$ such that ς_1 is not the conclusion of an application of a rule of type 2 other than the *falsum* rule, ς_n is not a minor premise of a application of a rule of type 2, and for all $i < n$, ς_i is minor premise of a rule of type 2. A segment consists of exactly one string which is not minor premise of an elimination rule of type two, or it consists of several strings which are minor premises of applications of rules of type two.

A string ς may be said to be *in* a segment σ and the formula on the string is also *on* the segment. A path is made up of a sequence of segments:

σ is a segment *of* a path π. Segments are premises/conclusions of applications of rules of inference depending on whether the last/first string in it is premise/conclusion of such a rule.

We may now prove the following theorem about the form of paths in deductions in normal form.

Theorem 9. Let Π be a deduction in normal form, π a path in Π and $\sigma_1 \dots \sigma_n$ the segments of π. There is a minimal segment σ_i such that, if it is not σ_n, it is premise of an introduction rule or of the *falsum* rule (*i.e.* the formula on it is $\bot$), and it divides π into two (possibly empty) parts:

1) the E-part, where, for $j < i$, σ_j is a major premise of an elimination rule and the formula on it has the formula on σ_{j+1} as a subformula;

2) the I-part, where, for $i < j < n$, σ_j is a premise of an introduction rule and the formula on it is a subformula of the formula on σ_{j+1}.

Proof. By definition, any string on a path is either the major premise of an elimination rule or a minor premise of a rule of type 2 or the last formula of the path. Thus applications of elimination rules to the formulas on the strings on π must precede the applications of introduction rules, for if there was an introduction rule occurring before an application of an elimination rule, this would give rise to a maximal segment or maximal formula, contrary to hypothesis that Π is in normal form. So let σ_i be the segment on which the first string on π occurs which is premise of an introduction rule or of the *falsum* rule, or, if there is no such segment, let it be σ_n. σ_i is the minimal segment. By what has just been said and the form of the deduction rules, it satisfies conditions 1) and 2).

2.5.6 Subformula Property and Separation

The *order of a path* $\pi = \varsigma_1 \dots \varsigma_n$ in a deduction is defined as follows: if ς_n is the conclusion of the deduction or the main premise $\Xi\ \overline{x}\, D_1 \dots D_n$ of an application of a rule of type two, if its subformula D_j has a corresponding discharged assumption without ancestors, then the order of the path is 0; a path is of order $n+1$ if it ends with a minor premise of a rule of form 1 the major premise of which is on a path of order n. A *main path* is a path is a path of order 0 the last formula of which is the conclusion of the deduction.

Every path π in a deduction Π determines a subdeduction of Π in an obvious way: namely the subdeduction determined by the consecution having the last formula on the last string on π in its succedent.

Let the *order of a (sub-)deduction* Π be the number of paths on it which are of order 0 and end in a major premise of an application of a rule of form 2.

Theorem 10 (Subformula Property)**.** Let Π be a deduction in normal form and $\Pi(X \vdash A)$. Then any formula occurring anywhere in Π is either a subformula of a formula on X or a subformula of A.

Proof by induction over the order of deductions. Obviously, if a deduction is in normal form, so are all subdeductions determined by any consecution occurring in it. To establish the basis of the induction, let the order of a deduction Π of $X \vdash A$ be 0, *i.e.* any path of order 0 is a main path in Π.

We first prove by induction over the order of paths that any formula on a path of Π is either a subformula of A or of a formula on X:

Let π be a path of order 0. By Theorem 9, for any path in a deduction, the formulas on segments on the E-part of a path and on the minimal segment are subformulas of the formula highest up in the path, and the formulas on the segments of the I-part are subformulas of the last formula of the path. By hypothesis, Π is a main path. So if there is an application of a rule of form two in the E-part, π goes from a hypothesis through the major premise all the way down through a collateral deduction of the rule to the conclusion. So any formula on π is either a subformula of A, or of the first formula on π, which is introduced into the deduction by an initial consecution. The formula of the same shape in its antecedent either is the last formula on a branch ending in an discharged assumption or it is not. In the latter case, obviously it is on X. In the former case, it can only have been discharged by an application of an introduction rule of form one, the conclusion of which must occur in the I-part of π. Hence it is a subformula of A. It could not have been discharged by an application of a rule of form two, because then it would not be the first formula of π. Hence any formula on a path of order 0 in Π is either a subformula of A or of X. Assume that this holds for paths order k. Then it must hold for paths π_n of order $n > k$. The last formula C of π_n is a subformula of a formula on a path of order $n-1$, so it is either a subformula of a formula on X or of A, by hypothesis. Any formula on π_n is either on its E-part, the minimal segment, or the I-part. All formulas in the I-part are subformulas of C, so it follows that they, too, are subformulas of A or of a formula on X. The formulas on the E-part and on the minimal segment are subformulas of the first formula on π_n, which has been introduced by an initial consecution. If this formula is not discharged in the deduction, then it is on X. If it is, then it is either subformula of a formula on π_n, if some premises of the rule which discharge it occur in it, or somewhere lower in the deduction, if this is not the case, *i.e.* it is subformula of a path with order less than n, so that by hypothesis it is a subformula of X or of A. Thus any formula on a path of Π is either a subformula of A or of a formula on X.

This leaves, secondly, the formulas in antecedents of consecutions to be considered. That they too are either subformulas of A or of a formula of X can easily be seen, by recalling that the only way for a formula to get into a deduction at all, and thus *a fortiori* into the antecedent of consecutions, is to be introduced by an initial segment or by Thinning. These are either discharged or not. In the first case, they appear on some path of the deduction, in the other case they occur on X, so that in either case, what was to be proved holds.

This completes the proof of the basis of the induction.

By induction hypothesis, for any deduction Π of $X \vdash A$ of order less than k, any formula occurring anywhere in Π is either a subformula of a formula on X or a subformula of A. We need to show that this holds for subdeductions of order $n > k$.

So suppose Π has order $n > k$. Take the major premise of an application of a rule of form 2 in Π in which a path of order 0 ends such that no other formula of the same kind occurs below it in the deduction. The last step of the subdeduction Π' determined by the conclusion of the rule is of the form:

$$\frac{Z \vdash \Xi\,\overline{x}\,D_1 \ldots D_n \quad Y(\Phi(D_h \ldots D_i)) \vdash E \quad \ldots \quad Y(\Psi(D_l \ldots D_m)) \vdash E}{Y(Z) \vdash E}$$

The subdeductions leading to the premises of the rules are of order lower than n, so by hypothesis, the theorem holds for them. To show that the theorem holds for the subdeduction leading to $Y(Z) \vdash E$ consider the following. All formulas occurring in collateral deductions, except the discharged assumptions, re-occur in the conclusion of the deduction, and the discharged premises are subformulas of $\Xi\overline{x}D_1 \ldots D_n$, and furthermore the list on which it depends, too, re-occurs in the conclusion of the rule. $\Xi\overline{x}D_1 \ldots D_n$ must be on the E-part of paths going through or ending in it, for otherwise it would be a maximal formula, contrary to hypothesis that Π is in normal form. Applying the induction hypothesis to Π', it follows that $\Xi\overline{x}D_1 \ldots D_n$ is either a subformula of a formula on Z or of the conclusion of Π', *i.e.* it is a subformula of itself. In the latter case, it must also be a subformula of a formula on Z, more precisely, Z consists only of $\Xi\overline{x}D_1 \ldots D_n$, as then this formula must have been introduced by an initial consecution: inspection of the rules shows that there is no way of introducing a formula as an initial consecution, then applying only elimination rules with it as the major premise, and eventually regaining the same formula. (Recall that $\Xi\overline{x}D_1 \ldots D_n$ must be on the E-part of paths). As all the discharged assumptions of the elimination rule of type two which is the last step of Π' are subformulas of $\Xi\overline{x}D_1 \ldots D_n$, it follows that any formula on the subdeduction Π' determined by $Y(Z) \vdash E$ is either a subformula of E or of a formula on $Y(Z)$. By Lemma 3, E is not a premise of an elimination rule. Hence it is in the I-part of a path, which accordingly, given how Π' has been chosen, either ends in the conclusion of Π or in a minor premise of an application of a rule of type 1. In the first case, by the form of introduction rules, E is a subformula of the conclusion of Π', and the formulas on $Y(Z)$ either re-occur in the antecedent X of the conclusion A of Π or they are subformulas of A. In the other case, if $Y(Z) \vdash E$ is on a path of order $n+1$, then the formulas on $Y(Z)$ are either on the list the last formula of the path depends on or subformulas of this last formula, and E is also a subformula of it. But then the E is subformula of a formula on a path of order n and the formulas on $Y(Z)$ are on a list on which a formula on a path of order n depends. Induction shows they are subformulas of X or of A.

Q.e.d.

Corollary 11 (Separation)**.** Let Π be a deduction in normal form and $\Pi(X \vdash A)$. The only operational rules applied in Π are rules for connectives that occur in formulas on X or in A.

Proof. Immediate from Theorem 10 and the forms of operational rules.

2.5.7 Conservative Extensions

Corollary 11 provides the basis for an easy proof of the conservativeness of connectives governed by rules of type 1 and 2. Let L be a quasi-intuitionist logic with language $\mathcal{L}$ and deductive system $\mathfrak{R}$ determining a consequence relation $\vdash$. Let L_Ξ be L extended by a connective Ξ, *i.e.* $\mathcal{L}_\Xi$ is an extension of $\mathcal{L}$ by Ξ and $\mathfrak{R}_\Xi$ is $\mathfrak{R}$ extended by rules of type one or two for Ξ, determining a consequence relation $\vdash_{\Xi}$.

Theorem 12 (Conservativeness)**.** $\vdash_{\Xi}$ is a conservative extension of $\vdash$: if X and A are in the vocabulary of L, then $X \vdash A$ iff $X \vdash_{\Xi} A$.

Proof. $\Rightarrow$ Trivial: $\vdash_{\Xi}$ is an extension of $\vdash$. $\Leftarrow$ Assume $X \vdash_{\Xi} A$. Then by Theorem 4 there is a deduction in normal form of A from X. Now X and A are in the vocabulary of L. By Corollary 11, this deduction cannot use rules for the connective that extends L to L_{Ξ}. Hence $X \vdash A$.

2.5.8 Cut-Elimination

Consider the following one-place connective δ governed by rules of type two:

$$\frac{X \vdash \delta A \qquad Y(A) \vdash C}{Y(X) \vdash C}$$

$$\frac{Z \vdash A}{Z \vdash \delta A}$$

δ could be interpreted as 'It is true that'. Replacing δA by A in δE gives the Cut-Rule:

$$\frac{X \vdash A \qquad Y(A) \vdash C}{Y(X) \vdash C}$$

Obviously, $A \dashv\vdash \delta A$. By theorem 12, δ may be added conservatively to any quasi-intuitionist logic. Assume the Cut-Rule is added to such a logic. Then any use of the Cut-Rule can be avoided in deductions by inserting an application of δI to its left premise and then replacing the Cut-Rule by an application of δE. The converse, of course, holds too, and any use of δE

may be replaced by an application of the Cut-Rule, if δA is systematically replaced by A in the deduction (making applications of δI vacuous and thus redundant). The local peaks resulting from replacing Cut by applications of δI and δE may be levelled by applying the reduction procedures. This, then, also gives a method for eliminating applications of Cut. Hence any use of the Cut-Rule can be eliminated from deductions and adding Cut produces a conservative extension. We record this result in the following theorem, which establishes a very close connection between Cut-Elimination and Normalisation in the present framework:

Theorem 13. Cut is an admissible rule in any regular logic.

2.5.9 The Adequacy Problem

Quasi-intuitionist logics have been allowed to have any number of connectives, governed by any kind of rules of type one or type two. But clearly there is no need for, *e.g.*, an undefined connective for a three-place conjunction or a three-place disjunction, as these could be defined in terms of $\wedge$ and $\vee$ as $((A \wedge B) \wedge C)$ and $((A \vee B) \vee C)$, respectively, if these are part of the logic in question. Also, there is no need for a primitive connective governed by the rules

$$\frac{(X; A_1), A_2 \vdash B_1 \qquad X, (A_3; A_4) \vdash B_2}{X \vdash \Xi A_1 A_2 A_3 A_4 B_1 B_2}\ \Xi I$$

$$\frac{Z \vdash \Xi A_1 A_2 A_3 A_4 B_1 B_2 \qquad Y_1 \vdash A_1 \qquad Y_2 \vdash A_2}{(Z; Y_1), Y_2 \vdash B_1}\ \Xi E_1$$

$$\frac{Z \vdash \Xi A_1 A_2 A_3 A_4 B_1 B_2 \qquad Y_3 \vdash A_3 \qquad Y_4 \vdash A_4}{Z, (Y_3; Y_4) \vdash B_2}\ \Xi E_2$$

If the logic has implications and conjunctions for the comma and the semi-colon (ignoring the possibility of left and right versions thereof), say $\rightarrow$, $\times$, $\supset$ and $\wedge$, then $\Xi A_1A_2A_3A_4B_1B_2$ may be defined as $(A_1 \rightarrow (A_2 \supset B_1)) \wedge ((A_3 \times A_4) \supset B_4)$.

Similarly, there is no need for a connective governed by the rules

$$\frac{X_1 \vdash D_1[x/t] \qquad X_2 \vdash D_2[x/t]}{X_1; X_2 \vdash \Xi x D_1D_2D_3}\ \Xi I_1 \qquad\qquad \frac{X_3 \vdash D_3[x/t]}{X_3 \vdash \Xi x D_1D_2D_3}\ \Xi I_2$$

$$\frac{Z \vdash \Xi x D_1D_2D_3 \qquad Y(D_1[x/a]; D_2[x/a]) \vdash E \qquad Y(D_3[x/a]) \vdash E}{Y(Z) \vdash E}\ \Xi E$$

where a does not occur in E and any formula on Y.

If the logic has disjunction $\vee$, conjunction $\times$ for the semi-colon and existential quantification $\exists$, then $\Xi x D_1D_2D_3$ may be defined as $\exists x((D_1 \times D_2) \vee D_3)$.

Generalising these observations, n-place quantifiers, for $n > 1$, governed by rules of type one are universally quantified conjunctions of implications for groupings with conjunctions for groupings in their antecedents, and n-place quantifiers, $n > 1$, governed by rules of type two are existentially quantified disjunctions of conjunctions for groupings. This leaves the cases where $n = 0$ and $n = 1$. They are covered by *verum*, *falsum*, the truth constant for the empty list **t** and the truth operators **T** and δ. The latter are again definable, for instance $\mathbf{T}A =_{def.} A\&A$ and $\delta A =_{def.} A \vee A$.[20]

[20] *Verum* cannot be definable as an arbitrary theorem if a logic does not have Thinning. Sometimes *verum* and *falsum* are explained as the disjunction and conjunction, respectively, of all formulas, but from what has just been said, *verum* may with the same right be considered to be the empty disjunction and *falsum* the empty conjunction, *i.e.* with no formulas to be disjoined or conjoined. In fact, of course, *verum* and *falsum* are neither of these, but are zero-place connectives.

This discussion extends to the present framework, and thereby modifies at important points, what Zucker and Tragesser call *the adequacy problem for inferential logic*.[21] This is the problem of showing that a set of connectives is adequate for a logic L formulated in a natural deduction system in the sense that every logical operation of L may be defined explicitly in terms of these connectives, if the meanings of the connective are supposed to be given by inference rules. Zucker and Tragesser consider only deductions made from collections of formulas for which the structural rules of intuitionist or classical logic hold. No such restrictions arise in the present framework. A restriction that needs to be made is that the rules are *stable*, not merely *harmonious*; Zucker and Tragesser implicitly respect a similar restriction for their framework.

Theorem 14. If a quasi-intuitionist logic L contains only stable rules, then the set of connectives containing *verum* $\top$, *falsum* $\bot$, conjunction $\wedge$, disjunction $\vee$, implications and conjunctions for each grouping of L (left and right, if they differ), a truth constant for the empty list 0, universal quantification and existential quantification is adequate for L.

If a logic fulfils the condition of the theorem, it is in a clear sense *complete*, so that theorem 14 provides a completeness theorem for a formal system relative to proof-theoretic semantics. In some cases, *e.g.* intuitionist logic, some of the connectives mentioned in the theorem are redundant, so that an even smaller set is adequate.

2.5.10 Restricting the *falsum* Rule to Atomics

Theorem 15. *Ex falso quodlibet* can be restricted to atomic conclusions, if $\Theta(\zeta_1/X \ldots \zeta_n/X) \Leftarrow X$ is an admissible structural rule for all structures

[21] Zucker, J.I. & Tragesser, R.S.: 'The Adequacy Problem for Inferential Logic', *Journal of Philosophical Logic* 7 (1978), 501-516.

$\Theta(\zeta_1 \ldots \zeta_n)$ occurring in collateral deductions in rules of type two governing a constant Ξ in a logic L.

Proof. First, it holds quite generally for any logic that a constant governed by rules of type one need never be the main operator of a conclusion of *ex falso quodlibet.* For suppose a step of the last kind occurs in a deduction:

$$\dfrac{\begin{matrix}\Pi \\ X \vdash \bot\end{matrix}}{Z(X) \vdash \Xi\overline{x}A_1 \ldots A_k B_1 \ldots B_p[\overline{a}/\overline{x}]} \\ \Sigma$$

This may be replaced by the following construction:

$$\dfrac{\dfrac{\begin{matrix}\Pi \\ X \vdash \bot\end{matrix}}{\Phi(X\ A_1 \ldots A_h) \vdash B_1} \quad \ldots \quad \dfrac{\begin{matrix}\Pi \\ X \vdash \bot\end{matrix}}{\Psi(X\ A_k \ldots A_l) \vdash B_p}}{X \vdash \Xi\,\overline{x}\,A_1 \ldots A_j B_1 \ldots B_p} \\ \Sigma$$

Secondly, consider the case where the main premise of the conclusion of *ex falso quodlibet* is governed by rules of type two:

$$\dfrac{\begin{matrix}\Pi \\ X \vdash \bot\end{matrix}}{Z(X) \vdash \Xi\,\overline{x}\,D_1 \ldots D_n} \\ \Sigma$$

Suppose the premises required for ΞI are derived by *ex falso quodlibet* with $Z(X)$ in their antecedent, before the rule is applied:

$$
\dfrac{\dfrac{\begin{array}{c}\Pi \\ X \vdash \bot\end{array}}{Z(X) \vdash D_j[\overline{x}/\overline{a}]} \quad \ldots \quad \dfrac{\begin{array}{c}\Pi \\ X \vdash \bot\end{array}}{Z(X) \vdash D_k[\overline{x}/\overline{a}]}}{\Theta(Z(X) \ldots Z(X)) \vdash \Xi\, \overline{x}\, D_1 \ldots D_n}
$$

Then if the condition of the theorem is satisfied by the structural rules of the logic, adding the relevant steps P by structural rules gives the construction that shows that *ex falso quodlibet* may be restricted to atomic formulas in the logic:

$$
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\Pi \\ X \vdash \bot\end{array}}{Z(X) \vdash D_j} \quad \ldots \quad \dfrac{\begin{array}{c}\Pi \\ X \vdash \bot\end{array}}{Z(X) \vdash D_k}}{\Theta(Z(X) \ldots Z(X) \vdash \Xi\, \overline{x}\, D_1 \ldots D_n} \\
\mathrm{P} \\
Z(X) \vdash \Xi\, \overline{x}\, D_1 \ldots D_n \\
\Sigma
\end{array}
$$

2.6 Quasi-Intuitionist Relevant Logics

Quasi-intuitionist relevant logics have in addition to rules of type one and type two also the introduction and elimination rules for negation using **f** introduced in section 2.2.4:

$$
\dfrac{\Theta(X\ A) \vdash \mathbf{f}}{X \vdash \neg A} \qquad \dfrac{Z \vdash \neg A \qquad Y \vdash A}{\Theta(Z\ Y) \vdash \mathbf{f}}
$$

It is also required that Thinning is not an admissible structural rule for the salient grouping of Θ: if it was, the irrelevant *ex contradictione quodlibet* is provable, as $X \vdash A$ is derivable from $X \vdash \mathbf{f}$ by Thinning and $\neg$I.

Local peaks with negation can be levelled:

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \Theta(X\ A)\vdash \mathbf{f}\end{array}}{X\vdash\neg A}\qquad \begin{array}{c}\Sigma\\ Y\vdash A\end{array}}{\begin{array}{c}\Theta(X\ Y)\vdash\mathbf{f}\\ \mathrm{P}\end{array}}$$

can be replaced by

$$\begin{array}{c}\Sigma\\ Y\vdash A\\ =\,=\,=\,=\,=\\ \Pi[A/Y]\\ \Theta(X\ Y)\vdash\mathbf{f}\\ \mathrm{P}\end{array}$$

The lemmata, theorems and corrolaries of section 2.5 go through also for quasi-intuitionist relevant logics. To see this, notice first that $\mathbf{f}$ can only ever occur in deductions of quasi-intuitionist relevant logics in places where $\bot$ occurs in quasi-intuitionist logics: any occurrence of $\neg A$ can be replaced by an occurrence of $A \rightarrow \bot$, for a suitable implication connective. Then ¬I becomes an application of →I and ¬E becomes an application of →E. Thus, the deductions of quasi-intuitionist relevant logics correspond to deductions of quasi-intuitionist logics that contain no application of *ex falso quodlibet*. For consistency, 8 goes through unchanged, and for there to be a proof of $\mathbf{f}$, i.e. $0 \vdash \mathbf{f}$, there would have to be a proof of A and of $\neg A$ from 0, but the first is impossible, by 8. As $\mathbf{f}$ is not a formula, theorem 10 goes through unchanged.

2.7 Quasi-Classical Logics

Quasi-classical logics are those regular logics which have, in addition to ¬I and ¬E of section 2.6, the negation rule *consequentia mirabilis*:

$$\dfrac{\Theta(X\ A)\vdash f}{X\vdash\neg A}\qquad\qquad \dfrac{X\vdash\neg A\qquad Y\vdash A}{\Theta(X\ Y)\vdash f}$$

$$\frac{\Theta(X\ \neg A) \vdash f}{X \vdash A}$$

$\neg$ may also be called a *negation for* the grouping salient in Θ. Notice that once more there are two options for a negation for a grouping γ which may not amount to the same if $X;Y \Leftarrow Y;X$ is not a structural rule for ;.

The definition of *local peak* and *maximal string* needs to be amended to cover also the cases a string is conclusion of *consequentia mirabilis* and major premise for an elimination rule for the main connective $*$ of the formula on it. Let's call this kind of local peak *local peaks with consequentia mirabilis and* $*$. As in the case of quasi-intuitionist relevant logics, f is not considered to be governed by any rules at all, so it is never the formula on a maximal string. The reduction procedure for local peaks with $\neg$ are the same as those in the last section. Giving reduction procedures for local peaks with *consequentia mirabilis* requires some more information about the groupings and structural rules of the logic. Consider a local peak with *consequentia mirabilis* and $\rightarrow$:

$$\cfrac{\cfrac{\begin{matrix}\Pi \\ \Theta(X\ \neg(A \rightarrow B)) \vdash f\end{matrix}}{X \vdash A \rightarrow B} \qquad \begin{matrix}\Sigma \\ Y \vdash A\end{matrix}}{\Phi(X\ Y) \vdash B}$$

In the absence of more information about the groupings of Θ and Φ, we don't get anywhere. So let's assume that $\Theta(\xi_1\ \xi_2) = \Phi(\xi_1\ \xi_2) = \xi_1;\xi_2$. This still isn't enough. We need to know whether we can transform Σ into something suitable to append to Π and make replacements so as to transform it into a deduction of B from $X;Y$. Suppose the structural rules Associativity $X;(Y;Z) \Leftarrow (X;Y);Z$ and Permutation $X;Y \Leftarrow Y;X$ are available. Then the following construction would achieve the aim of levelling the local peak:

$$
\dfrac{\dfrac{A \to B \vdash A \to B \qquad \begin{array}{c}\Sigma\\ Y \vdash A\end{array}}{A \to B; Y \vdash B} \qquad \neg B \vdash \neg B}{\dfrac{\dfrac{\dfrac{\dfrac{\neg B; (A \to B; Y) \vdash f}{\neg B; (Y; A \to B) \vdash f}}{(\neg B; Y); A \to B \vdash f}}{\neg B; Y \vdash \neg(A \to B)}}{Y; \neg B \vdash \neg(A \to B)}}
$$

$$
\begin{array}{c}
= = = = = = = = = = = = = =\\
\Pi[\neg(A \to B)/Y; \neg B]\\
\dfrac{\dfrac{X; (Y; \neg B) \vdash f}{(X; Y); \neg B \vdash f}}{X; Y \vdash B}
\end{array}
$$

Obviously, if the conclusions of *consequentia mirabilis* can be restricted to atomic formulas, no local peaks with *consequentia mirabilis* and another constant can arise. But whether this is possible or not depends, once more, on the further details of the logic. Consequently, rather than proving general theorems about quasi-classical logics, in the present section I shall consider the two systems **C** and **R** individually.

The necessity of treating them differently shows an interesting contrast between quasi-classical and quasi-intuitionist logic—the latter can be discussed in a very general fashion, and specific systems subsumed under large classes of logics which are shown to be proof-theoretically justified, whereas the former need to be considered almost on a case to case bass. Generalisations are of course possible here, too, but they fall short of the generality possible for quasi-intuitionist logics. This difference does not establish any philosophical advantage of quasi-intuitionist and quasi-classical logics, however. The proof-theoretic justification of deduction requires only that a logic be justified, not that it be part of a larger class of justified logics.

2.7.1 Classical Logic

The structural rules for the comma are Associativity $X,(Y,Z) \Leftarrow (X,Y),Z$, Contraction $X,X \Leftarrow X$, Permutation $X,Y \Leftarrow Y,Z$ and Thinning $X \Leftarrow Y,X$, and for the empty list Left Addition $X \Leftarrow 0,X$ and Left Subtraction $X,0 \Leftarrow X$. The logical constants are &, $\neg$ and $\forall$. The conclusion of *consequentia mirabilis* can be restricted to atomic formulas. This is no loss, as any constant governed by rules of type two can be defined in terms of negation and constants governed by rules of type one. The *falsum* rule, with $\bot$ replaced by **f**, is derivable by applying the negation and structural rules.

Because the reduction procedures for local peaks with $\neg$ are so similar to the reduction procedures for local peaks with implication, lemma 2 obviously still goes through in classical logic, and so does theorem 4, as using *consequentia mirabilis* with conclusions restricted to atomics cannot lead to local peaks with *consequentia mirabilis*.

The proof of theorem 5 requires the addition of a clause in the proof of the basis of the induction, namely that if only one rule is applied in the deduction, it cannot be one of the negation rules. The additional clauses required in the proof of the induction step are in part parallel to the considerations about rules of type one: the last step of a proof in classical logic cannot be one by application of $\neg$E, for reasons similar to those that show that the last step cannot be one by $\supset$E. The clause needed to show that the last step cannot be one by *consequentia mirabilis* can be proved by referring to theorem 5 for intuitionist logic. If the last step in the deduction is by *consequentia mirabilis*, then the deduction ends thus:

$$\cfrac{\begin{array}{c}\Pi\\ 0, \neg p \vdash f\end{array}}{0 \vdash p}$$

If any formula p which is the conclusion of *consequentia mirabilis* is replaced by $\neg p$ throughout the deduction, then the deduction is intuitionistically valid, as any application of *consequentia mirabilis* can then be replaced by applications of rules of intuitionist logic. Hence, if the last step of a proof in classical logic could be *consequentia mirabilis*, *i.e.* p would be a theorem of classical logic, then $\neg p$ would be a theorem of intuitionist logic, contradicting the consequence of corollary 8 that this cannot be. Hence the last step of a proof in classical logic cannot be by *consequentia mirabilis*. This proves Theorem 5 for classical logic with *consequentia mirabilis* restricted to atomic conclusions, and its corollaries 6, 7 and 8 follow, too.

A normalisation theorem can also be proved for the case where *consequentia mirabilis* is not restricted to atomic formulas and rules of form two are present. Any local peak with *consequentia mirabilis* and another constant can be levelled, but I won't go into the details.

The subformula property doesn't hold for our system of classical logic: the proof of $\neg\neg A \vdash A$ is a counterexample:

$$\dfrac{\dfrac{\neg\neg A \vdash \neg\neg A \qquad \neg A \vdash \neg A}{\neg\neg A, \neg A \vdash \mathbf{f}}}{\neg\neg A \vdash A}$$

However, a restricted subformula property hold. If we consider $\mathbf{f}$ not to be a formula, we need to allow for formulas of the form $\neg A$ to be amongst the hypotheses of a deduction or introduced by Thinning, which are not subformulas of the conclusion $X \vdash B$ of the deduction. These will have been carried down some subdeduction of the deduction until they are discharged by an application of *consequentia mirabilis* (which has as its premise $\mathbf{f}$, which, if we did consider it to be a formula, would also not be a subformula of any formula of $X \vdash B$).

Separation still holds and so does 5. These results are, however, less 'stable' as in the case of quasi-intuitionist logics, as adding, for instance, the connective $\supset$ entails the theorems fail.

We can also say that $\forall$, & and $\neg$ are adequate for classical logic, as any other connectives that might be added to it can be defined in terms of it and its rules constructed as derived rules of inference.

2.7.2 Relevance Logic

The situation with relevance logic in the proof-theoretic justification of deduction is more complicated and merits a paper on its own right. I'll only add some brief considerations here.

Consider the system of logic which has a grouping ; for which the structural rules are Associativity $X; (Y; Z) \Leftarrow (X; Y); Z$, Contraction $X; X \Leftarrow X$, Permutation $X; Y \Leftarrow Y; Z$, left addition for the semi-colon and the empty list $X \Leftarrow t; X$ and left subtraction $t; X \Leftarrow X$, and which has an implication $\rightarrow$, a quasi-classical negation $\neg$ for this grouping and & as connectives. *Consequentia mirabilis* can be restricted to atomic conclusions. The normalisation theorem for this logic goes through as it does for classical logic, so it counts as proof-theoreticall justified according to my approach. It has the subformula property as in classical logic and separation holds. It is functionally complete in the sense that all rules of type one and type two can be constructed from the primitive rules. We can define a conjunction $\times$ for ; as $\neg(A \rightarrow \neg B)$ for which the rules on p.46 hold and we can define a disjunction $\vee$ as $\neg(\neg A \& \neg B)$, which has the rules on p.47. The proofs relies on the presence of the structural rules Associativity and Permutation, which allow the re-ordering of assumptions.

In this logic, Distribution $A\&(B \vee C) \vdash (A\&B) \vee C$ does not hold. From the perspective of the proof-theoretic justification of deduction, this logic, i.e. **R** without Distribution, is straightforwardly justified, which may be a bit surprising, given Distribution is such an intuitive principle. Some philosophers might be delighted to hear, then, that the current account, although more liberal than Dummett's original account, potentially retains some of Dummett's revisionism.

But that depends on what can be said about the full system **R**. To get the full system **R** with Distribution, we could add the principle as an additional rule. But this is not a rule of either type one or type two. Alternatively, we could add the comma as a second grouping with the structural rules Associativity $X,(Y,Z) \Leftarrow (X,Y),Z$, Contraction $X,X \Leftarrow X$, Permutation $X,Y \Leftarrow Y,Z$ and Thinning $X \Leftarrow Y,X$ (and none for the comma and the empty list), and then add $\vee$ governed by the rules of p.2.2.3 as a new primitive (and also add a conjunction $\times$ for the semi-colon as a primitive). In this system, however, I have not found a way of levelling local peaks with *consequentia mirabilis* and $\vee$, although I have no proof that it's impossible. The adequacy problem is more complicated in the case of relevance logic, because of the presence of two groupings. These issues merit a paper of its own right, so I shall not attempt to tackle them here.

2.8 Dummett's Conjecture

Recall Dummett's Conjecture: 'intrinsic harmony implies total harmony in a context where stability prevails' (*LBM* 290). As argued, Dummett's notions of intrinsic harmony and stability seem, on the one hand, to be intended to apply to rules of inference independently of logics they may be part of, as characterising the forms of rules, and at the same time Dummett cashes them out formally in terms of normalisation, which is a notion applicable only to formal systems as a whole. The 'context where stability prevails' Dummett could refer either to a formal system under consideration or to the rules of a constant under consideration.

The conjecture also doesn't specify whether it is required that the logic resulting from adding a constant normalises or not. We can assume that, as Dummett is interested in the proof-theoretic justification of deduction, he assumes the resulting logic also to normalise. Harmony entails stability on my definitions, so it suffices to consider the weaker case of harmonious rules.

One way of reading Dummett's Conjecture is this one: Adding a constant with harmonious rules to a logic with stable rules produces a conservative extension. That's false, on my account of harmony and stability: adding classical negation (with *consequentia mirabilis* not restricted to atomic formulas) to intuitionist logic does not produce a conservative extension. The logic also normalises, as local peaks with *consequentia mirabilis* and $\vee$ as well as $\exists$ can be levelled.

However, Dummmett's Conjecture holds if restricted to quasi-inuitionist logics. If Ξ is governed by harmonious rules, then adding it to a quasi-intuitionist logic with only constants governed by harmonious rules, the deductions of which normalise, produces a conservative extension thereof, if the resulting logic also normalises. This is true, as the resulting logic is also quasi-intuitionist. The condition that the resulting logic normalises excludes, for instance, the case of adding intuitionst disjunction to quantum logic: this logic does not normalise.

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